3.2 Conditional Probability and the Multiplication Rule

What You SHOULD LEARN

- How to find the probability of an event given that another event has occurred
- How to distinguish between independent and dependent events
- How to use the Multiplication Rule to find the probability of two events occurring in sequence
- How to use the Multiplication Rule to find conditional probabilities

Conditional Probability ▸ Independent and Dependent Events ▸ The Multiplication Rule

▸ Conditional Probability

In this section, you will learn how to find the probability that two events occur in sequence. Before you can find this probability, however, you must know how to find conditional probabilities.

DEFINITION

A **conditional probability** is the probability of an event occurring, given that another event has already occurred. The conditional probability of event B occurring, given that event A has occurred, is denoted by $P(B|A)$ and is read as "probability of B, given A."

EXAMPLE 1

Finding Conditional Probabilities

1. Two cards are selected in sequence from a standard deck. Find the probability that the second card is a queen, given that the first card is a king. (Assume that the king is not replaced.)
2. The table at the left shows the results of a study in which researchers examined a child's IQ and the presence of a specific gene in the child. Find the probability that a child has a high IQ, given that the child has the gene. (*Source: Psychological Science*)

	Gene present	Gene not present	Total
High IQ	33	19	52
Normal IQ	39	11	50
Total	72	30	102

Solution

1. Because the first card is a king and is not replaced, the remaining deck has 51 cards, 4 of which are queens. So,

$$P(B|A) = \frac{4}{51} \approx 0.078.$$

So, the probability that the second card is a queen, given that the first card is a king, is about 0.078.

2. There are 72 children who have the gene. So, the sample space consists of these 72 children, as shown at the left. Of these, 33 have a high IQ. So,

$$P(B|A) = \frac{33}{72} \approx 0.458.$$

So, the probability that a child has a high IQ, given that the child has the gene, is about 0.458.

Sample Space

	Gene present
High IQ	33
Normal IQ	39
Total	72

▸ Try It Yourself 1

1. Find the probability that a child does not have the gene.
2. Find the probability that a child does not have the gene, given that the child has a normal IQ.

a. Find the *number of outcomes* in the event and in the sample space.
b. *Divide* the number of outcomes in the event by the number of outcomes in the sample space.

Answer: Page A37

Independent and Dependent Events

In some experiments, one event does not affect the probability of another. For instance, if you roll a die and flip a coin, the outcome of the roll of the die does not affect the probability of the coin landing on heads. These two events are independent. The question of the independence of two or more events is important to researchers in fields such as marketing, medicine, and psychology. You can use conditional probabilities to determine whether events are independent.

DEFINITION

Two events are **independent** if the occurrence of one of the events does not affect the probability of the occurrence of the other event. Two events A and B are independent if

$$P(B|A) = P(B) \quad \text{or if} \quad P(A|B) = P(A).$$

Events that are not independent are **dependent.**

To determine if A and B are independent, first calculate $P(B)$, the probability of event B. Then calculate $P(B|A)$, the probability of B, given A. If the values are equal, the events are independent. If $P(B) \neq P(B|A)$, then A and B are dependent events.

PICTURING the WORLD

Truman Collins, a probability and statistics enthusiast, wrote a program that finds the probability of landing on each square of a Monopoly board during a game. Collins explored various scenarios, including the effects of the Chance and Community Chest cards and the various ways of landing in or getting out of jail. Interestingly, Collins discovered that the length of each jail term affects the probabilities.

Monopoly square	Probability given short jail term	Probability given long jail term
Go	0.0310	0.0291
Chance	0.0087	0.0082
In Jail	0.0395	0.0946
Free Parking	0.0288	0.0283
Park Place	0.0219	0.0206
B&B RR	0.0307	0.0289
Water Works	0.0281	0.0265

Why do the probabilities depend on how long you stay in jail?

EXAMPLE 2

Classifying Events as Independent or Dependent

Decide whether the events are independent or dependent.

1. Selecting a king from a standard deck (A), not replacing it, and then selecting a queen from the deck (B)
2. Tossing a coin and getting a head (A), and then rolling a six-sided die and obtaining a 6 (B)
3. Driving over 85 miles per hour (A), and then getting in a car accident (B)

Solution

1. $P(B|A) = \frac{4}{51}$ and $P(B) = \frac{4}{52}$. The occurrence of A changes the probability of the occurrence of B, so the events are dependent.
2. $P(B|A) = \frac{1}{6}$ and $P(B) = \frac{1}{6}$. The occurrence of A does not change the probability of the occurrence of B, so the events are independent.
3. If you drive over 85 miles per hour, the chances of getting in a car accident are greatly increased, so these events are dependent.

Try It Yourself 2

Decide whether the events are independent or dependent.

1. Smoking a pack of cigarettes per day (A) and developing emphysema, a chronic lung disease (B)
2. Exercising frequently (A) and having a 4.0 grade point average (B)

a. *Decide* whether the occurrence of the first event affects the probability of the second event.
b. State if the events are *independent* or *dependent*. *Answer: Page A38*

The Multiplication Rule

To find the probability of two events occurring in sequence, you can use the Multiplication Rule.

Study Tip

In words, to use the Multiplication Rule,

1. find the probability the first event occurs,
2. find the probability the second event occurs given the first event has occurred, and
3. multiply these two probabilities.

THE MULTIPLICATION RULE FOR THE PROBABILITY OF A AND B

The probability that two events A and B will occur in sequence is

$$P(A \text{ and } B) = P(A) \cdot P(B|A).$$

If events A and B are independent, then the rule can be simplified to $P(A \text{ and } B) = P(A) \cdot P(B)$. This simplified rule can be extended for any number of independent events.

EXAMPLE 3

Using the Multiplication Rule to Find Probabilities

1. Two cards are selected, without replacing the first card, from a standard deck. Find the probability of selecting a king and then selecting a queen.
2. A coin is tossed and a die is rolled. Find the probability of getting a head and then rolling a 6.

Solution

1. Because the first card is not replaced, the events are dependent.

$$\begin{aligned} P(K \text{ and } Q) &= P(K) \cdot P(Q|K) \\ &= \frac{4}{52} \cdot \frac{4}{51} \\ &= \frac{16}{2652} \\ &\approx 0.006 \end{aligned}$$

So, the probability of selecting a king and then a queen is about 0.006.

2. The events are independent.

$$\begin{aligned} P(H \text{ and } 6) &= P(H) \cdot P(6) \\ &= \frac{1}{2} \cdot \frac{1}{6} \\ &= \frac{1}{12} \\ &\approx 0.083 \end{aligned}$$

So, the probability of tossing a head and then rolling a 6 is about 0.083.

Try It Yourself 3

1. The probability that a salmon swims successfully through a dam is 0.85. Find the probability that two salmon successfully swim through the dam.
2. Two cards are selected from a standard deck without replacement. Find the probability that they are both hearts.

a. Decide if the events are *independent* or *dependent*.
b. Use the *Multiplication Rule* to find the probability.

Answer: Page A38

EXAMPLE 4

Using the Multiplication Rule to Find Probabilities

1. A coin is tossed and a die is rolled. Find the probability of getting a tail and then rolling a 2.
2. The probability that a particular knee surgery is successful is 0.85. Find the probability that three knee surgeries are successful.
3. Find the probability that none of the three knee surgeries is successful.
4. Find the probability that at least one of the three knee surgeries is successful.

Solution

1. $P(T) = \frac{1}{2}$. Whether or not the coin is a tail, $P(2) = \frac{1}{6}$. The events are independent.

$$P(T \text{and } 2) = P(T) \cdot P(2) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12} \approx 0.083$$

So, the probability of tossing a tail and then rolling a 2 is about 0.083.

2. The probability that each knee surgery is successful is 0.85. The chance of success for one surgery is independent of the chances for the other surgeries.

$$P(3 \text{ surgeries are successful}) = (0.85)(0.85)(0.85) \approx 0.614$$

So, the probability that all three surgeries are successful is about 0.614.

3. Because the probability of success for one surgery is 0.85, the probability of failure for one surgery is $1 - 0.85 = 0.15$.

$$P(\text{none of the three is successful}) = (0.15)(0.15)(0.15) \approx 0.003$$

So, the probability that none of the surgeries is successful is about 0.003.

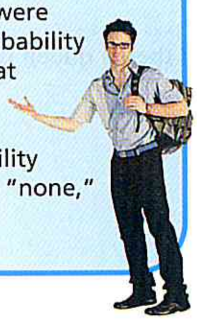
4. The phrase "at least one" means one or more. The complement to the event "at least one is successful" is the event "none are successful." Using the rule of complements,

$$P(\text{at least 1 is successful}) = 1 - P(\text{none are successful}) \approx 1 - 0.003 = 0.997.$$

There is about a 0.997 probability that at least one of the three surgeries is successful.

Insight

In Example 4, you were asked to find a probability using the phrase "at least one." Notice that it was easier to find the probability of its complement, "none," and then use the Complement Rule.



▶ Try It Yourself 4

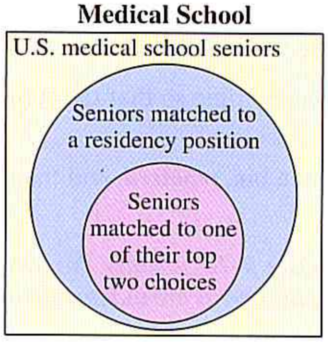
The probability that the knee surgery is successful has increased to 0.9.

1. Find the probability that three knee surgeries are successful.
2. Find the probability that at least one of three knee surgeries is successful..

a. Determine whether to find the probability of the event or its complement.
b. Use the *Multiplication Rule* to find the probability. If necessary, use the *Complement Rule*.

Answer: Page A38

Medical School



EXAMPLE 5

Using the Multiplication Rule to Find Probabilities

More than 15,000 U.S. medical school seniors applied to residency programs in 2007. Of those, 93% were matched to a residency position. Seventy-four percent of the seniors matched to a residency position were matched to one of their top two choices. Medical students electronically rank the residency programs in their order of preference and program directors across the United States do the same. The term "match" refers to the process where a student's preference list and a program director's preference list overlap, resulting in the placement of the student for a residency position. *(Source: National Resident Matching Program)*

1. Find the probability that a randomly selected senior was matched to a residency position *and* it was one of the senior's top two choices.
2. Find the probability that a randomly selected senior that was matched to a residency position did *not* get matched with one of the senior's top two choices.
3. Would it be unusual for a randomly selected senior to result in a senior that was matched to a residency position and it was one of the senior's top two choices?

Solution Let $A = \{\text{matched to residency position}\}$ and $B = \{\text{matched to one of two top choices}\}$. So, $P(A) = 0.93$ and $P(B|A) = 0.74$.

1. These events are dependent.

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = (0.93) \cdot (0.74) \approx 0.688$$

So, the probability that a randomly selected senior was matched to one of the top two choices is about 0.688.

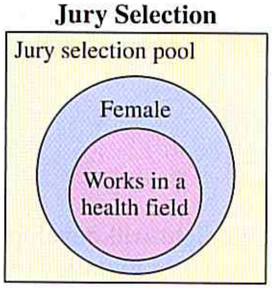
2. To find this probability, use the complement.

$$P(B'|A) = 1 - P(B|A) = 1 - 0.74 = 0.26$$

So, the probability that a randomly selected senior was matched to a residency position that was not one of the top two choices is 0.26.

3. It is not unusual because the probability of a senior being matched to a residency position and it was one of the top two choices is about 0.688.

Jury Selection



Try It Yourself 5

In a jury selection pool, 65% of the people are female. Of these 65%, one out of four works in a health field.

1. Find the probability that a randomly selected person from the jury pool is female and works in a health field.
2. Find the probability that a randomly selected person from the jury pool is female and does not work in a health field.

a. Determine *events A* and *B*.
b. Use the *Multiplication Rule* to write a formula to find the probability. If necessary, use the *Complement Rule*.
c. *Calculate* the probability.

Answer: Page A38

3.2 EXERCISES

Building Basic Skills and Vocabulary

1. What is the difference between independent and dependent events?
2. List examples of the following types of events.
   (a) Two events that are independent
   (b) Two events that are dependent

True or False? *In Exercises 3 and 4, determine whether the statement is true or false. If it is false, rewrite it as a true statement.*

3. If two events are not independent, $P(A|B) = P(B)$.
4. If events A and B are dependent, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

Classifying Events *In Exercises 5–8, decide whether the events are independent or dependent. Explain your reasoning.*

5. Selecting a king from a standard deck, replacing it, and then selecting a queen from the deck
6. Returning a rented movie after the due date and receiving a late fee
7. Rolling a six-sided die and then rolling the die a second time so that the sum of the two rolls is seven
8. A numbered ball between 1 and 52 is selected from a bin, *replaced*, and then a second numbered ball is selected from the bin.

Classifying Events Based on Studies *In Exercises 9–12, identify the two events described in the study. Do the results indicate that the events are independent or dependent? Explain your reasoning.*

9. Researchers found that people with depression are five times more likely to have a breathing-related sleep disorder than people who are not depressed. *(Source: Journal of Clinical Psychiatry)*
10. Stress causes the body to produce higher amounts of acid, which can irritate already existing ulcers. But, stress does not cause stomach ulcers. *(Source: Baylor College of Medicine)*
11. Studies found that Aspartame, an artificial sweetener, does not cause memory loss. *(Source: Food and Drug Administration)*
12. According to researchers, diabetes is rare in societies in which obesity is rare. In societies in which obesity has been common for at least 20 years, diabetes is also common. *(Source: American Diabetes Association)*

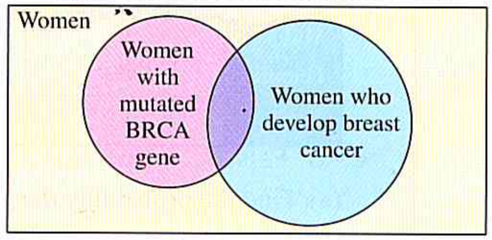
Using and Interpreting Concepts

13. **BRCA Gene** In the general population, one woman in eight will develop breast cancer. Research has shown that 1 woman in 600 carries a mutation of the BRCA gene. Eight out of 10 women with this mutation develop breast cancer. *(Source: Susan G. Komen Breast Cancer Foundation)*
    (a) Find the probability that a randomly selected woman will develop breast cancer given that she has a mutation of the BRCA gene.

(b) Find the probability that a randomly selected woman will carry the mutation of the BRCA gene and will develop breast cancer.

(c) Are the events of carrying this mutation and developing breast cancer independent or dependent? Explain.

Breast Cancer and the BRCA Gene

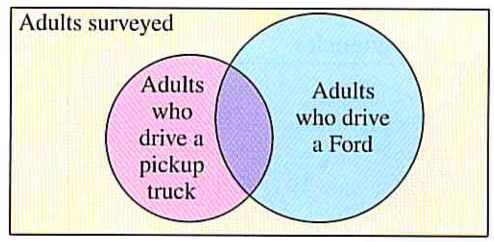


14. Pickup Trucks In a survey, 510 adults were asked if they drive a pickup truck and if they drive a Ford. The results showed that one in six adults surveyed drives a pickup truck, and three in ten adults surveyed drives a Ford. Of the adults surveyed that drive Fords, two in nine drive a pickup truck.

(a) Find the probability that a randomly selected adult drives a pickup truck given that he or she drives a Ford.

(b) Find the probability that a randomly selected adult drives a Ford and drives a pickup truck.

What Do You Drive?



(c) Are the events driving a Ford and driving a pickup truck independent or dependent? Explain.

15. Summer Vacation The table shows the results of a survey in which 146 families were asked if they own a computer and if they will be taking a summer vacation this year.

		Summer Vacation This Year		
		Yes	No	Total
Own a Computer	Yes	46	11	57
	No	55	34	89
	Total	101	45	146

(a) Find the probability that a randomly selected family is not taking a summer vacation this year.

(b) Find the probability that a randomly selected family owns a computer.

(c) Find the probability a randomly selected family is taking a summer vacation this year given that they own a computer.

(d) Find the probability a randomly selected family is taking a summer vacation this year and owns a computer.

(e) Are the events of owning a computer and taking a summer vacation this year independent or dependent events? Explain.

16. **Nursing Majors** The table shows the number of male and female students enrolled in nursing at the University of Oklahoma Health Sciences Center for a recent semester. *(Adapted from University of Oklahoma Health Center Office of Admissions and Records)*

	Nursing majors	Non-nursing majors	Total
Males	95	1015	1110
Females	700	1727	2427
Total	795	2742	3537

(a) Find the probability that a randomly selected student is a nursing major.

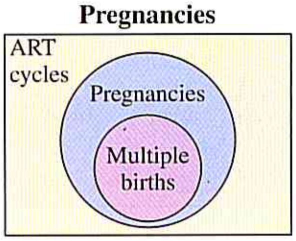
(b) Find the probability that a randomly selected student is male.

(c) Find the probability that a randomly selected student is a nursing major given that the student is male.

(d) Find the probability that a randomly selected student is a nursing major and male.

(e) Are the events being a male student and being a nursing major independent or dependent events? Explain.

Pregnancies



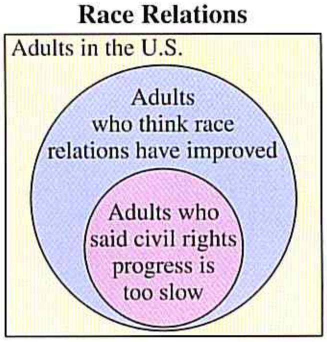
17. **Assisted Reproductive Technology** A study found that 35% of the assisted reproductive technology (ART) cycles resulted in a pregnancy. Twenty-eight percent of the ART pregnancies resulted in multiple births. *(Source: National Center for Chronic Disease Prevention and Health Promotion)*

(a) Find the probability that a randomly selected ART cycle resulted in a pregnancy *and* produced a multiple birth.

(b) Find the probability that a randomly selected ART cycle that resulted in a pregnancy did *not* produce a multiple birth.

(c) Would it be unusual for a randomly selected ART cycle to result in a pregnancy and produce a multiple birth? Explain.

Race Relations



18. **Race Relations** In a survey, 60% of adults in the United States think race relations have improved since the death of Martin Luther King Jr. Of these 60%, 4 out of 10 said the rate of civil rights progress is too slow. *(Source: Marist Institute for Public Opinion)*

(a) Find the probability that a randomly selected adult thinks race relations have improved since the death of Martin Luther King Jr. *and* thinks the rate of civil rights progress is too slow.

(b) Given that a randomly selected adult thinks race relations have improved since the death of Martin Luther King Jr., find the probability that he or she thinks the rate of civil rights progress is *not* too slow.

(c) Would it be unusual for a randomly selected adult to think race relations have improved since the death of Martin Luther King Jr. and think the rate of civil rights progress is too slow? Explain.

19. **Computers and Internet Access** A study found that 62% of households in the United States have a computer. Of those 62%, 88% have Internet access. Find the probability that a U.S. household selected at random has a computer and has Internet access. *(Source: U.S. Census Bureau)*

20. **Surviving Surgery** A doctor gives a patient a 60% chance of surviving bypass surgery after a heart attack. If the patient survives the surgery, he has a 50% chance that the heart damage will heal. Find the probability that the patient survives surgery and the heart damage heals.

21. Left-Handed People In a sample of 1000 people, 120 are left-handed. Two unrelated people are selected at random without replacement.

(a) Find the probability that both people are left-handed.

(b) Find the probability that neither person is left-handed.

(c) Find the probability that at least one of the two people is left-handed.

22. Light Bulbs Twelve light bulbs are tested to see if they last as long as the manufacturer claims they do. Three light bulbs fail the test. Two light bulbs are selected at random.without replacement.

(a) Find the probability that both light bulbs failed the test.

(b) Find the probability that both light bulbs passed the test.

(c) Find the probability that at least one light bulb failed the test.

23. Emergency Savings The table shows the results of a survey in which 142 men and 145 women workers ages 25 to 64 were asked if they have at least one month's income set aside for emergencies.

	Men	Women	Total
Less than one month's income	66	83	149
One month's income or more	76	62	138
Total	142	145	287

(a) Find the probability that a randomly selected worker has one month's income or more set aside for emergencies.

(b) Given that a randomly selected worker is a male, find the probability that the worker has less than one month's income.

(c) Given that a randomly selected worker has one month's income or more, find the probability that the worker is a female.

(d) Are the events "having less than one month's income saved" and "being male" independent or dependent? Explain.

24. Health Care for Dogs The table shows the results of a survey in which 90 dog owners were asked how much they have spent in the last year for their dog's health care, and whether their dogs were purebred or mixed breeds.

		Type of Dog		
		Purebred	Mixed breed	Total
Health Care	Less than $100	19	21	40
	$100 or more	35	15	50
	Total	54	36	90

(a) Find the probability that $100 or more was spent on a randomly selected dog's health care in the last year.

(b) Given that a randomly selected dog owner spent less than $100, find the probability that the dog was a mixed breed.

(c) Find the probability that a randomly selected dog owner spent $100 or more on health care and the dog was a mixed breed.

(d) Are the events "spending $100 or more on health care" and "having a mixed breed dog" independent or dependent? Explain.

25. Blood Types The probability that a person in the United States has type AB^+ blood is 3%. Five unrelated people in the United States are selected at random. *(Source: American Association of Blood Banks)*

(a) Find the probability that all five have type AB^+ blood.

(b) Find the probability that none of the five has type AB^+ blood.

(c) Find the probability that at least one of the five has type AB^+ blood.

26. Blood Types The probability that a person in the United States has type O^+ blood is 38%. Three unrelated people in the United States are selected at random. *(Source: American Association of Blood Banks)*

(a) Find the probability that all three have type O^+ blood.

(b) Find the probability that none of the three has type O^+ blood.

(c) Find the probability that at least one of the three has type O^+ blood.

27. Guessing A multiple-choice quiz has three questions, each with five answer choices. Only one of the choices is correct. You have no idea what the answer is to any question and have to guess each answer.

(a) Find the probability of answering the first question correctly.

(b) Find the probability of answering the first two questions correctly.

(c) Find the probability of answering all three questions correctly.

(d) Find the probability of answering none of the questions correctly.

(e) Find the probability of answering at least one of the questions correctly.

28. Bookbinding Defects A printing company's bookbinding machine has a probability of 0.005 of producing a defective book. This machine is used to bind three books.

(a) Find the probability that none of the books is defective.

(b) Find the probability that at least one of the books is defective.

(c) Find the probability that all of the books are defective.

29. Warehouses A distribution center receives shipments of a product from three different factories in the following quantities: 50, 35, and 25. Three times a product is selected at random, each time without a replacement. Find the probability that all three products came from the third factory.

30. Birthdays Three people are selected at random. Find the probability that (a) all three share the same birthday and (b) none of the three shares the same birthday. Assume 365 days in a year.

Extending Concepts

*According to **Bayes's Theorem,** the probability of event A, given that event B has occurred, is*

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(A') \cdot P(B|A')}.$$

In Exercises 31–34, use Bayes's Theorem to find $P(A|B)$.

31. $P(A) = \frac{2}{3}$, $P(A') = \frac{1}{3}$, $P(B|A) = \frac{1}{5}$, and $P(B|A') = \frac{1}{2}$

32. $P(A) = \frac{3}{8}$, $P(A') = \frac{5}{8}$, $P(B|A) = \frac{2}{3}$, and $P(B|A') = \frac{3}{5}$

33. $P(A) = 0.25$, $P(A') = 0.75$, $P(B|A) = 0.3$, and $P(B|A') = 0.5$

34. $P(A) = 0.62$, $P(A') = 0.38$, $P(B|A) = 0.41$, and $P(B|A') = 0.17$

35. Reliability of Testing A certain virus infects one in every 200 people. A test used to detect the virus in a person is positive 80% of the time if the person has the virus and 5% of the time if the person does not have the virus. (This 5% result is called a *false positive*.) Let A be the event "the person is infected" and B be the event "the person tests positive."

(a) Using Bayes's Theorem, if a person tests positive, determine the probability that the person is infected.

(b) Using Bayes's Theorem, if a person tests negative, determine the probability that the person is *not* infected.

36. Birthday Problem You are in a class that has 24 students. You want to find the probability that at least two of the students share the same birthday.

(a) First, find the probability that each student has a different birthday.

$$P(\text{different birthdays}) = \overbrace{\frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365}\cdot\frac{362}{365}\cdots\frac{343}{365}\cdot\frac{342}{365}}^{\text{24 factors}}$$

(b) The probability that at least two students have the same birthday is the complement of the probability in part (a). What is this probability?

(c) We used a technology tool to generate 24 random numbers between 1 and 365. Each number represents a birthday. Did we get at least two people with the same birthday?

228	348	181	317	81	183
52	346	177	118	315	273
252	168	281	266	285	13
118	360	8	193	57	107

(d) Use a technology tool to simulate the "Birthday Problem." Repeat the simulation 10 times. How many times did you get "at least two people" with the same birthday?

The Multiplication Rule and Conditional Probability *By rewriting the formula for the Multiplication Rule, you can write a formula for finding conditional probabilities. The conditional probability of event B occurring, given that event A has occurred, is*

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}.$$

In Exercises 37 and 38, use the following information.

- *The probability that an airplane flight departs on time is 0.89.*
- *The probability that a flight arrives on time is 0.87.*
- *The probability that a flight departs and arrives on time is 0.83.*

37. Find the probability that a flight departed on time given that it arrives on time.

38. Find the probability that a flight arrives on time given that it departed on time.