



(b)

For the receiving antenna shown in the figure, the common focus F is located 120 feet above the vertex of the parabola, and focus F' (for the hyperbola) is 20 feet above the vertex. The vertex of the reflecting hyperbola is 110 feet above the vertex for the parabola. Introduce a coordinate system by using the axis of the parabola as the y axis (up positive), and let the x axis pass through the center of the hyperbola (right positive). What is the equation of the reflecting hyperbola? Write y in terms of x .

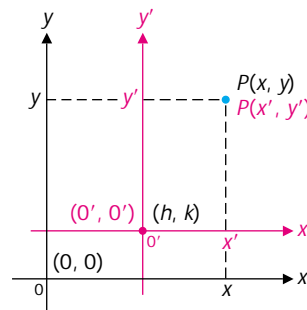
SECTION 9-4 Translation of Axes

- Translation of Axes
- Standard Equations of Translated Conics
- Graphing Equations of the Form $Ax^2 + Cy^2 + Dx + Ey + F = 0$
- Finding Equations of Conics

In the last three sections we found standard equations for parabolas, ellipses, and hyperbolas located with their axes on the coordinate axes and centered relative to the origin. What happens if we move conics away from the origin while keeping their axes parallel to the coordinate axes? We will show that we can obtain new standard equations that are special cases of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A and C are not both zero. The basic mathematical tool used in this endeavor is *translation of axes*. The usefulness of translation of axes is not limited to graphing conics, however. Translation of axes can be put to good use in many other graphing situations.

- **Translation of Axes** A **translation of coordinate axes** occurs when the new coordinate axes have the same direction as and are parallel to the original coordinate axes. To see how coordinates in the original system are changed when moving to the translated system, and vice versa, refer to Figure 1.

FIGURE 1 Translation of coordinates.



A point P in the plane has two sets of coordinates: (x, y) in the original system and (x', y') in the translated system. If the coordinates of the origin of the translated system are (h, k) relative to the original system, then the old and new coordinates are related as given in Theorem 1.

Theorem 1**Translation Formulas**

$$\begin{array}{ll} 1. & x = x' + h \\ & y = y' + k \end{array} \qquad \begin{array}{ll} 2. & x' = x - h \\ & y' = y - k \end{array}$$

It can be shown that these formulas hold for (h, k) located anywhere in the original coordinate system.

EXAMPLE 1 Equation of a Curve in a Translated System

A curve has the equation

$$(x - 4)^2 + (y + 1)^2 = 36$$

If the origin is translated to $(4, -1)$, find the equation of the curve in the translated system and identify the curve.

Solution Since $(h, k) = (4, -1)$, use translation formulas

$$x' = x - h = x - 4$$

$$y' = y - k = y + 1$$

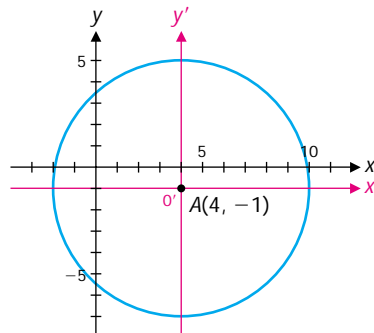
to obtain, after substitution,

$$x'^2 + y'^2 = 36$$

This is the equation of a circle of radius 6 with center at the new origin. The coordinates of the new origin in the original coordinate system are $(4, -1)$ (Fig. 2). Note that this result agrees with our general treatment of the circle in Section 3-1.

FIGURE 2

$$(x - 4)^2 + (y + 1)^2 = 36.$$



Matched Problem 1

A curve has the equation $(y + 2)^2 = 8(x - 3)$. If the origin is translated to $(3, -2)$, find an equation of the curve in the translated system and identify the curve.

- **Standard Equations of Translated Conics**

We now proceed to find standard equations of conics translated away from the origin. We do this by first writing the standard equations found in earlier sections in the $x'y'$ coordinate system with O' at (h, k) . We then use translation equations to find the standard forms relative to the original xy coordinate system. The equations of translation in all cases are

$$x' = x - h$$

$$y' = y - k$$

For parabolas we have

$$x'^2 = 4ay' \quad (x - h)^2 = 4a(y - k)$$

$$y'^2 = 4ax' \quad (y - k)^2 = 4a(x - h)$$

For circles we have

$$x'^2 + y'^2 = r^2 \quad (x - h)^2 + (y - k)^2 = r^2$$

For ellipses we have for $a > b > 0$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \quad \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\frac{x'^2}{b^2} + \frac{y'^2}{a^2} = 1 \quad \frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

For hyperbolas we have

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1 \quad \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

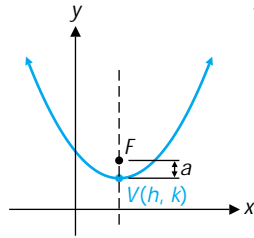
$$\frac{y'^2}{a^2} - \frac{x'^2}{b^2} = 1 \quad \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

Table 1 summarizes these results with appropriate figures and some properties discussed earlier.

TABLE 1 Standard Equations for Translated Conics

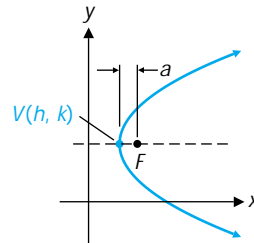
Parabolas

$$(x - h)^2 = 4a(y - k)$$



Vertex (h, k)
 Focus $(h, k + a)$
 $a > 0$ opens up
 $a < 0$ opens down

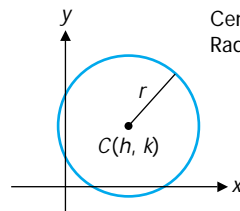
$$(y - k)^2 = 4a(x - h)$$



Vertex (h, k)
 Focus $(h + a, k)$
 $a < 0$ opens left
 $a > 0$ opens right

Circles

$$(x - h)^2 + (y - k)^2 = r^2$$

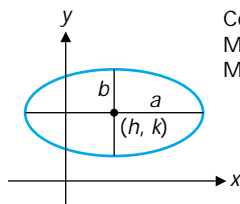


Center (h, k)
 Radius r

Ellipses

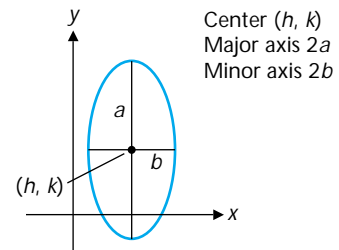
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$a > b > 0$$



Center (h, k)
 Major axis $2a$
 Minor axis $2b$

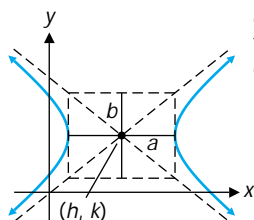
$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$



Center (h, k)
 Major axis $2a$
 Minor axis $2b$

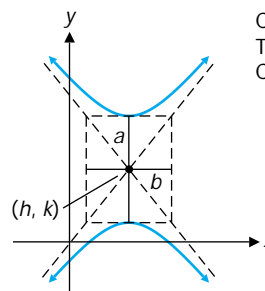
Hyperbolas

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$



Center (h, k)
 Transverse axis $2a$
 Conjugate axis $2b$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$



Center (h, k)
 Transverse axis $2a$
 Conjugate axis $2b$

• Graphing
Equations of the
Form $Ax^2 + Cy^2 + Dx$
 $+ Ey + F = 0$

It can be shown that the graph of

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \quad (1)$$

where A and C are not both zero, is a conic or a degenerate conic or that there is no graph. If we can transform equation (1) into one of the standard forms in Table 1, then we will be able to identify its graph and sketch it rather quickly. The process of completing the square discussed in Section 2-6 will be our primary tool in accomplishing this transformation. A couple of examples should help make the process clear.

EXAMPLE 2 Graphing a Translated Conic

Transform

$$y^2 - 6y - 4x + 1 = 0 \quad (2)$$

into one of the standard forms in Table 1. Identify the conic and graph it.

Solution *Step 1.* Complete the square in equation (2) relative to each variable that is squared—in this case y :

$$\begin{aligned} y^2 - 6y - 4x + 1 &= 0 \\ y^2 - 6y &= 4x - 1 \\ y^2 - 6y + 9 &= 4x + 8 && \text{Add 9 to both sides to complete} \\ &&& \text{the square on the left side.} \\ (y - 3)^2 &= 4(x + 2) \end{aligned} \quad (3)$$

From Table 1 we recognize equation (3) as an equation of a parabola opening to the right with vertex at $(h, k) = (-2, 3)$.

Step 2. Find the equation of the parabola in the translated system with origin O' at $(h, k) = (-2, 3)$. The equations of translation are read directly from equation (3):

$$\begin{aligned} x' &= x + 2 \\ y' &= y - 3 \end{aligned}$$

Making these substitutions in equation (3) we obtain

$$y'^2 = 4x' \quad (4)$$

the equation of the parabola in the $x'y'$ system.

Step 3. Graph equation (4) in the $x'y'$ system following the process discussed in Section 9-1. The resulting graph is the graph of the original equation relative to the original xy coordinate system (Fig. 3).

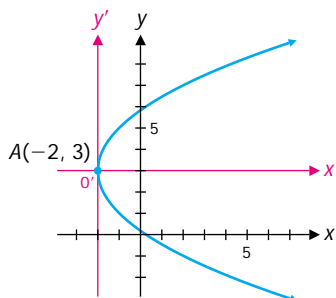


FIGURE 3
 $y^2 - 6y - 4x + 1 = 0$.

Matched Problem 2 Transform

$$x^2 + 4x + 4y - 12 = 0$$

into one of the standard forms in Table 1. Identify the conic and graph it.

EXAMPLE 3 Graphing a Translated Conic

Transform

$$9x^2 - 4y^2 - 36x - 24y - 36 = 0$$

into one of the standard forms in Table 1. Identify the conic and graph it. Find the coordinates of any foci relative to the original system.

Solution *Step 1.* Complete the square relative to both x and y .

$$\begin{aligned} 9x^2 - 4y^2 - 36x - 24y - 36 &= 0 \\ 9x^2 - 36x \quad - 4y^2 - 24y &= 36 \\ 9(x^2 - 4x \quad) - 4(y^2 + 6y \quad) &= 36 \\ 9(x^2 - 4x + 4) - 4(y^2 + 6y + 9) &= 36 + 36 - 36 \\ 9(x - 2)^2 - 4(y + 3)^2 &= 36 \\ \frac{(x - 2)^2}{4} - \frac{(y + 3)^2}{9} &= 1 \end{aligned}$$

From Table 1 we recognize the last equation as an equation of a hyperbola opening left and right with center at $(h, k) = (2, -3)$.

Step 2. Find the equation of the hyperbola in the translated system with origin O' at $(h, k) = (2, -3)$. The equations of translation are read directly from the last equation in step 1:

$$\begin{aligned} x' &= x - 2 \\ y' &= y + 3 \end{aligned}$$

Making these substitutions, we obtain

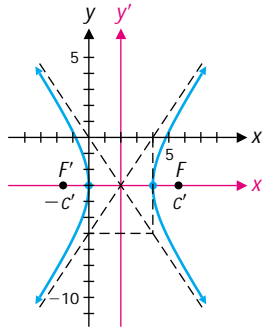
$$\frac{x'^2}{4} - \frac{y'^2}{9} = 1$$

the equation of the hyperbola in the $x'y'$ system.

Step 3. Graph the equation obtained in step 2 in the $x'y'$ system following the process discussed in Section 9-3. The resulting graph is the graph of the original equation relative to the original xy coordinate system (Fig. 4).

FIGURE 4

$$9x^2 - 4y^2 - 36x - 24y - 36 = 0.$$



Step 4. Find the coordinates of the foci. To find the coordinates of the foci in the original system, first find the coordinates in the translated system:

$$c'^2 = 2^2 + 3^2 = 13$$

$$c' = \sqrt{13}$$

$$-c' = -\sqrt{13}$$

Thus, the coordinates in the translated system are

$$F'(-\sqrt{13}, 0) \quad \text{and} \quad F(\sqrt{13}, 0)$$

Now, use

$$x = x' + h = x' + 2$$

$$y = y' + k = y' - 3$$

to obtain

$$F'(-\sqrt{13} + 2, -3) \quad \text{and} \quad F(\sqrt{13} + 2, -3)$$

as the coordinates of the foci in the original system.

Matched Problem 3 Transform

$$9x^2 + 16y^2 + 36x - 32y - 92 = 0$$

into one of the standard forms in Table 1. Identify the conic and graph it. Find the coordinates of any foci relative to the original system.



Remark. A graphing utility provides an alternative approach to graphing equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$. Consider, for example, the equation $9x^2 - 4y^2 - 36x - 24y - 36 = 0$ of Example 3. We write the equation as a quadratic equation in the variable y : $4y^2 + 24y + (-9x^2 + 36x + 36) = 0$. By the

quadratic formula, $y = \frac{-24 \pm \sqrt{24^2 - 16f(x)}}{8}$, where $f(x) = -9x^2 + 36x + 36$.

We then graph each of the two functions in the expression for y . The graph of $y = \frac{-24 + \sqrt{24^2 - 16f(x)}}{8}$ is the upper half of the hyperbola, and the graph of

$y = \frac{-24 - \sqrt{24^2 - 16f(x)}}{8}$ is the lower half.

EXPLORE-DISCUSS 1 If $A \neq 0$ and $C \neq 0$, show that the translation of axes $x' = x + \frac{D}{2A}$, $y' = y + \frac{E}{2C}$ transforms the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into an equation of the form $Ax'^2 + Cy'^2 = K$.

• **Finding Equations of Conics**

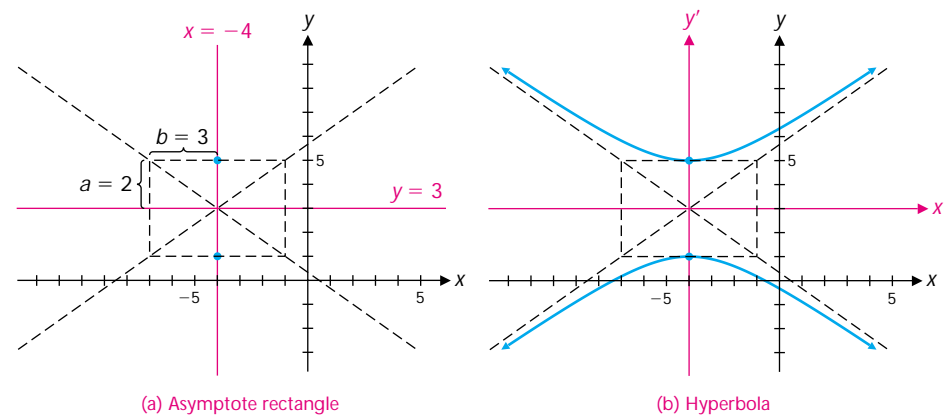
We now reverse the problem: Given certain information about a conic in a rectangular coordinate system, find its equation.

EXAMPLE 4 Finding the Equation of a Translated Conic

Find the equation of a hyperbola with vertices on the line $x = -4$, conjugate axis on the line $y = 3$, length of the transverse axis = 4, and length of the conjugate axis = 6.

Solution Locate the vertices, asymptote rectangle, and asymptotes in the original coordinate system [Fig. 5(a)], then sketch the hyperbola and translate the origin to the center of the hyperbola [Fig. 5(b)].

FIGURE 5



Next write the equation of the hyperbola in the translated system:

$$\frac{y'^2}{4} - \frac{x'^2}{9} = 1$$

The origin in the translated system is at $(h, k) = (-4, 3)$, and the translation formulas are

$$x' = x - h = x - (-4) = x + 4$$

$$y' = y - k = y - 3$$

Thus, the equation of the hyperbola in the original system is

$$\frac{(y - 3)^2}{4} - \frac{(x + 4)^2}{9} = 1$$

or, after simplifying and writing in the form of equation (1),

$$4x^2 - 9y^2 + 32x + 54y + 19 = 0$$

Matched Problem 4 Find the equation of an ellipse with foci on the line $x = 4$, minor axis on the line $y = -3$, length of the major axis = 8, and length of the minor axis = 4.

EXPLORE-DISCUSS 2 Use the strategy of completing the square to transform each equation to an equation in an $x'y'$ coordinate system. Note that the equation you obtain is not one of the standard forms in Table 1; instead, it is either the equation of a degenerate conic or the equation has no solution. If the solution set of the equation is not empty, graph it and identify the graph (a point, a line, two parallel lines, or two intersecting lines).

(A) $x^2 + 2y^2 - 2x + 16y + 33 = 0$

(B) $4x^2 - y^2 - 24x - 2y + 35 = 0$

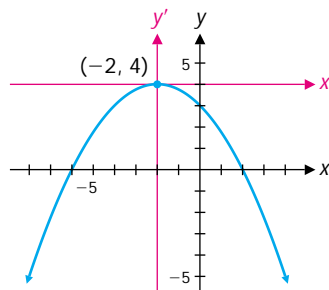
(C) $y^2 - 2y - 15 = 0$

(D) $5x^2 + y^2 + 12y + 40 = 0$

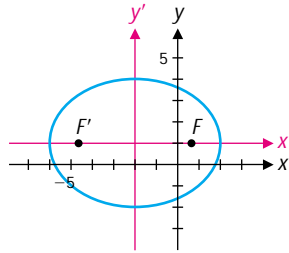
(E) $x^2 - 18x + 81 = 0$

Answers to Matched Problems

- $y'^2 = 8x'$; a parabola
- $(x + 2)^2 = -4(y - 4)$; a parabola



$$3. \frac{(x+2)^2}{16} + \frac{(y-1)^2}{9} = 1; \text{ ellipse} \quad \text{Foci: } F'(-\sqrt{7}-2, 1), F(\sqrt{7}-2, 1)$$



$$4. \frac{(x-4)^2}{4} + \frac{(y+3)^2}{16} = 1, \text{ or } 4x^2 + y^2 - 32x + 6y + 57 = 0$$

EXERCISE 9-4

A

In Problems 1–8:

- (A) Find translation formulas that translate the origin to the indicated point (h, k) .
 (B) Write the equation of the curve for the translated system.
 (C) Identify the curve.

- $(x-3)^2 + (y-5)^2 = 81; (3, 5)$
- $(x-3)^2 = 8(y+2); (3, -2)$
- $\frac{(x+7)^2}{9} + \frac{(y-4)^2}{16} = 1; (-7, 4)$
- $(x+2)^2 + (y+6)^2 = 36; (-2, -6)$
- $(y+9)^2 = 16(x-4); (4, -9)$
- $\frac{(y-9)^2}{10} - \frac{(x+5)^2}{6} = 1; (-5, 9)$
- $\frac{(x+8)^2}{12} + \frac{(y+3)^2}{8} = 1; (-8, -3)$
- $\frac{(x+7)^2}{25} - \frac{(y-8)^2}{50} = 1; (-7, 8)$

In Problems 9–14:

- (A) Write each equation in one of the standard forms listed in Table 1.
 (B) Identify the curve.
- $16(x-3)^2 - 9(y+2)^2 = 144$
 - $(y+2)^2 - 12(x-3) = 0$
 - $6(x+5)^2 + 5(y+7)^2 = 30$
 - $12(y-5)^2 - 8(x-3)^2 = 24$
 - $(x+6)^2 + 24(y-4) = 0$
 - $4(x-7)^2 + 7(y-3)^2 = 28$

B

In Problems 15–22, transform each equation into one of the standard forms in Table 1. Identify the curve and graph it.

- $4x^2 + 9y^2 - 16x - 36y + 16 = 0$
- $16x^2 + 9y^2 + 64x + 54y + 1 = 0$
- $x^2 + 8x + 8y = 0$
- $y^2 + 12x + 4y - 32 = 0$
- $x^2 + y^2 + 12x + 10y + 45 = 0$
- $x^2 + y^2 - 8x - 6y = 0$
- $-9x^2 + 16y^2 - 72x - 96y - 144 = 0$
- $16x^2 - 25y^2 - 160x = 0$
- If $A \neq 0$, $C = 0$, and $E \neq 0$, find h and k so that the translation of axes $x = x' + h$, $y = y' + k$ transforms the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into one of the standard forms of Table 1.
- If $A = 0$, $C \neq 0$, and $D \neq 0$, find h and k so that the translation of axes $x = x' + h$, $y = y' + k$ transforms the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into one of the standard forms of Table 1.

In Problems 25–34, use the given information to find the equation of each conic. Express the answer in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ with integer coefficients and $A \geq 0$.

- A parabola with vertex at $(5, 3)$, and focus at $(5, 11)$.
- A parabola with focus at $(2, 3)$, and directrix the y axis.
- An ellipse with vertices $(-3, -2)$ and $(-3, 10)$ and length of minor axis = 10.
- A hyperbola with vertices $(-2, -8)$ and $(4, -8)$ and length of conjugate axis = 24.

29. A hyperbola with foci (2, 1) and (6, 1) and vertices (3, 1) and (5, 1).
30. An ellipse with foci (-3, 0) and (-3, 6) and vertices (-3, -2) and (-3, 8).
31. A parabola with axis the y axis and passing through the points (1, 0) and (2, 4).
32. A parabola with vertex at (-6, 2), axis the line $y = 2$, and passing through the point (0, 7).
33. An ellipse with vertices (1, -1), and (-5, -1) that passes through the origin.
34. A hyperbola with vertices at (2, 3), and (2, 5) that passes through the point (4, 0).

C

In Problems 35–40, find the coordinates of any foci relative to the original coordinate system:

35. Problem 15 36. Problem 16 37. Problem 17
38. Problem 18 39. Problem 21 40. Problem 22



In Problems 41–44, use a graphing utility to find the coordinates of all points of intersection to two decimal places.

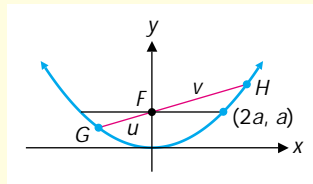
41. $3x^2 - 5y^2 + 7x - 2y + 11 = 0$, $6x + 4y = 15$
42. $8x^2 + 3y^2 - 14x + 17y - 39 = 0$, $5x - 11y = 23$
43. $7x^2 - 8x + 5y - 25 = 0$, $x^2 + 4y^2 + 4x - y - 12 = 0$
44. $4x^2 - y^2 - 24x - 2y + 35 = 0$, $2x^2 + 6y^2 - 3x - 34 = 0$

CHAPTER 9 GROUP ACTIVITY Focal Chords

Many of the applications of the conic sections are based on their reflective or focal properties. One of the interesting algebraic properties of the conic sections concerns their focal chords.

If a line through a focus F contains two points G and H of a conic section, then the line segment GH is called a **focal chord**. Let $G(x_1, y_1)$ and $H(x_2, y_2)$ be points on the graph of $x^2 = 4ay$ such that GH is a focal chord. Let u denote the length of GF and v the length of FH (see Fig. 1).

FIGURE 1 Focal chord GH of the parabola $x^2 = 4ay$.



- (A) Use the distance formula to show that $u = y_1 + a$.
- (B) Show that G and H lie on the line $y - a = mx$, where $m = (y_2 - y_1)/(x_2 - x_1)$.
- (C) Solve $y - a = mx$ for x and substitute in $x^2 = 4ay$, obtaining a quadratic equation in y . Explain why $y_1 y_2 = a^2$.
- (D) Show that $\frac{1}{u} + \frac{1}{v} = \frac{1}{a}$.
- (E) Show that $u + v - 4a = \frac{(u - 2a)^2}{u - a}$. Explain why this implies that $u + v \geq 4a$, with equality if and only if $u = v = 2a$.
- (F) Which focal chord is the shortest? Is there a longest focal chord?
- (G) Is $\frac{1}{u} + \frac{1}{v}$ a constant for focal chords of the ellipse? For focal chords of the hyperbola? Obtain evidence for your answers by considering specific examples.