2.7 Solving Equations in One Variable

What you'll learn about

- Solving Rational Equations
- Extraneous Solutions
- Applications

... and why

Applications involving rational functions as models often require that an equation involving the model be solved.

Solving Rational Equations

Equations involving rational expressions or fractions (see Appendix A.3) are **rational equations**. Every rational equation can be written in the form

$$\frac{f(x)}{g(x)} = 0.$$

If $f(x)$ and $g(x)$ are polynomial functions with no common factors, then the zeros of $f(x)$ are the solutions of the equation.

Usually it is not necessary to put a rational equation in the form of $f(x)/g(x)$. To solve an equation involving fractional expressions we begin by finding the LCD (least common denominator) of all the terms of the equation. Then we clear the equation of fractions by multiplying each side of the equation by the LCD. Sometimes the LCD contains variables.

When we multiply or divide an equation by an expression containing variables, the resulting equation may have solutions that are *not* solutions of the original equation. These are **extraneous solutions**. For this reason we must check each solution of the resulting equation in the original equation.

EXAMPLE 1 Solving by Clearing Fractions

Solve $x + \frac{3}{x} = 4$.

SOLUTION

Solve Algebraically

The LCD is x.

$$x + \frac{3}{x} = 4$$

$$x^2 + 3 = 4x \qquad \text{Multiply by } x.$$

$$x^2 - 4x + 3 = 0 \qquad \text{Subtract } 4x.$$

$$(x - 1)(x - 3) = 0 \qquad \text{Factor.}$$

$$x - 1 = 0 \quad \text{or} \quad x - 3 = 0 \qquad \text{Zero factor property}$$

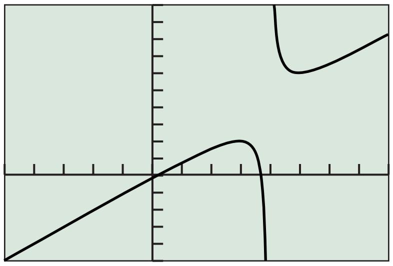
$$x = 1 \quad \text{or} \quad x = 3$$

Confirm Numerically

For $x = 1$, $x + \frac{3}{x} = 1 + \frac{3}{1} = 4$, and for $x = 3$, $x + \frac{3}{x} = 3 + \frac{3}{3} = 4$.

Each value is a solution of the original equation. ***Now try Exercise 1.***

When the fractions in Example 2 are cleared, we obtain a quadratic equation.



[–5, 8] by [–5, 10]

FIGURE 2.56 The graph of $y = x + 1/(x - 4)$. (Example 2)

EXAMPLE 2 Solving a Rational Equation

Solve $x + \dfrac{1}{x-4} = 0$.

SOLUTION

Solve Algebraically The LCD is $x - 4$.

$$x + \frac{1}{x-4} = 0$$

$$x(x-4) + \frac{x-4}{x-4} = 0 \qquad \text{Multiply by } x - 4.$$

$$x^2 - 4x + 1 = 0 \qquad \text{Distributive property}$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \qquad \text{Quadratic formula}$$

$$x = \frac{4 \pm 2\sqrt{3}}{2} \qquad \text{Simplify.}$$

$$x = 2 \pm \sqrt{3} \qquad \text{Simplify.}$$

$$x \approx 0.268, 3.732$$

Support Graphically

The graph in Figure 2.56 suggests that the function $y = x + 1/(x - 4)$ has two zeros. We can use the graph to find that the zeros are about 0.268 and 3.732, agreeing with the values found algebraically. ***Now try Exercise 7.***

Extraneous Solutions

We will find extraneous solutions in Examples 3 and 4.

EXAMPLE 3 Eliminating Extraneous Solutions

Solve the equation

$$\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{x^2 - 4x + 3}.$$

SOLUTION

Solve Algebraically

The denominator of the right-hand side, $x^2 - 4x + 3$, factors into $(x - 1)(x - 3)$. So the least common denominator (LCD) of the equation is $(x - 1)(x - 3)$, and we multiply both sides of the equation by this LCD:

$$(x-1)(x-3)\left(\frac{2x}{x-1} + \frac{1}{x-3}\right) = (x-1)(x-3)\left(\frac{2}{x^2-4x+3}\right)$$

$$2x(x-3) + (x-1) = 2 \qquad \text{Distributive property}$$

$$2x^2 - 5x - 3 = 0 \qquad \text{Distributive property}$$

$$(2x+1)(x-3) = 0 \qquad \text{Factor.}$$

$$x = -\frac{1}{2} \quad \text{or} \quad x = 3$$

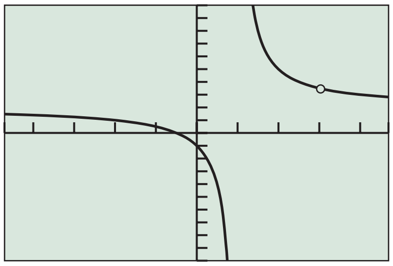
Confirm Numerically

We replace x by $-1/2$ in the original equation:

$$\frac{2(-1/2)}{(-1/2) - 1} + \frac{1}{(-1/2) - 3} \stackrel{?}{=} \frac{2}{(-1/2)^2 - 4(-1/2) + 3}$$

$$\frac{2}{3} - \frac{2}{7} \stackrel{?}{=} \frac{8}{21}$$

(continued)



[–4.7, 4.7] by [–10, 10]

FIGURE 2.57 The graph of $f(x) = 2x/(x - 1) + 1/(x - 3) - 2/(x^2 - 4x + 3)$, with the missing point at $(3, 3.5)$ emphasized. (Example 3)

The equation is true, so $x = -1/2$ is a valid solution. The original equation is not defined for $x = 3$, so $x = 3$ is an extraneous solution.

Support Graphically

The graph of

$$f(x) = \frac{2x}{x - 1} + \frac{1}{x - 3} - \frac{2}{x^2 - 4x + 3}$$

in Figure 2.57 suggests that $x = -1/2$ is an x-intercept and $x = 3$ is not.

Interpret

Only $x = -1/2$ is a solution. ***Now try Exercise 13.***

We will see that Example 4 has no solutions.

EXAMPLE 4 Eliminating Extraneous Solutions

Solve

$$\frac{x - 3}{x} + \frac{3}{x + 2} + \frac{6}{x^2 + 2x} = 0.$$

SOLUTION The LCD is $x(x + 2)$.

$$\frac{x - 3}{x} + \frac{3}{x + 2} + \frac{6}{x^2 + 2x} = 0$$

$$(x - 3)(x + 2) + 3x + 6 = 0 \qquad \text{Multiply by } x(x + 2).$$

$$x^2 - x - 6 + 3x + 6 = 0 \qquad \text{Expand.}$$

$$x^2 + 2x = 0 \qquad \text{Simplify.}$$

$$x(x + 2) = 0 \qquad \text{Factor.}$$

$$x = 0 \quad \text{or} \quad x = -2$$

Substituting $x = 0$ or $x = -2$ into the original equation results in division by zero. So both of these numbers are extraneous solutions and the original equation has no solution. ***Now try Exercise 17.***

Applications

EXAMPLE 5 Calculating Acid Mixtures

How much pure acid must be added to 50 mL of a 35% acid solution to produce a mixture that is 75% acid? (See Figure 2.58.)

SOLUTION

Model

Word statement: $\frac{\text{mL of pure acid}}{\text{mL of mixture}} = \text{concentration of acid}$

0.35×50 or $17.5 =$ mL of pure acid in 35% solution

$x =$ mL of acid added

$x + 17.5 =$ mL of pure acid in resulting mixture

$x + 50 =$ mL of the resulting mixture

$\frac{x + 17.5}{x + 50} =$ concentration of acid

Solve Graphically

$$\frac{x + 17.5}{x + 50} = 0.75 \qquad \text{Equation to be solved}$$

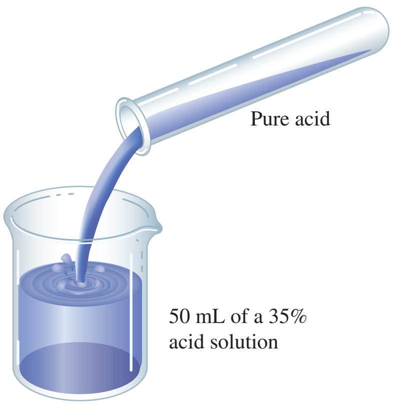


FIGURE 2.58 Mixing solutions. (Example 5)

Figure 2.59 shows graphs of $f(x) = (x + 17.5)/(x + 50)$ and $g(x) = 0.75$. The point of intersection is (80, 0.75).

Interpret

We need to add 80 mL of pure acid to the 35% acid solution to make a solution that is 75% acid. ***Now try Exercise 31.***

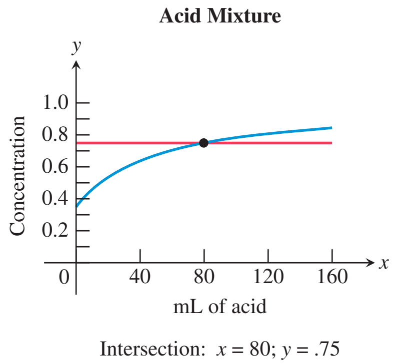


FIGURE 2.59 The graphs of $f(x) = (x + 17.5)/(x + 50)$ and $g(x) = 0.75$. (Example 5)

EXAMPLE 6 Finding a Minimum Perimeter

Find the dimensions of the rectangle with minimum perimeter if its area is 200 square meters. Find this least perimeter.

SOLUTION

Model

Draw the diagram in Figure 2.60.

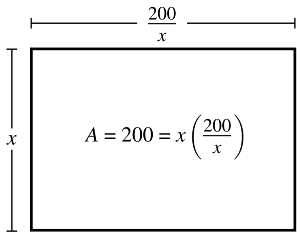


FIGURE 2.60 A rectangle with area 200 m^2. (Example 6)

Word statement: Perimeter = 2 × length + 2 × width

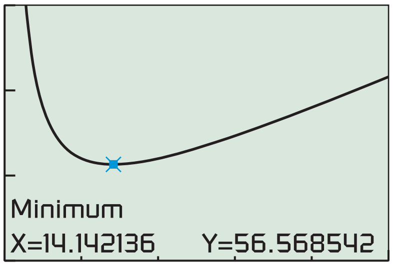
x = width in meters

$$\frac{200}{x} = \frac{\text{area}}{\text{width}} = \text{length in meters}$$

Function to be minimized: $P(x) = 2x + 2\left(\frac{200}{x}\right) = 2x + \frac{400}{x}$

Solve Graphically

The graph of P in Figure 2.61 shows a minimum of approximately 56.57, occurring when $x \approx 14.14$.



[0, 50] by [0, 150]

FIGURE 2.61 A graph of $P(x) = 2x + 400/x$. (Example 6)

Interpret

A width of about 14.14 m produces the minimum perimeter of about 56.57 m. Because $200/14.14 \approx 14.14$, the dimensions of the rectangle with minimum perimeter are 14.14 m by 14.14 m, a square. ***Now try Exercise 35.***

EXAMPLE 7 Designing a Juice Can

Stewart Cannery packages tomato juice in 2-liter cylindrical cans. Find the radius and height of the cans if the cans have a surface area of 1000 cm^2. (See Figure 2.62.)

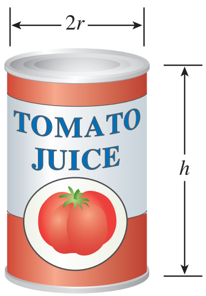


FIGURE 2.62 A tomato juice can. (Example 7)

SOLUTION

Model

S = surface area of can in cm^2

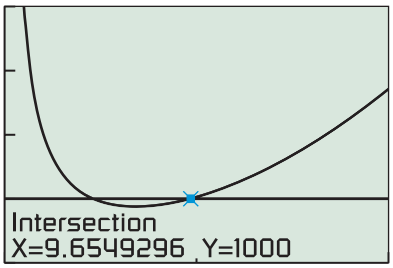
r = radius of can in centimeters

h = height of can in centimeters

Using volume (V) and surface area (S) formulas and the fact that 1 L = 1000 cm^3, we conclude that

$$V = \pi r^2 h = 2000 \quad \text{and} \quad S = 2\pi r^2 + 2\pi r h = 1000.$$

(continued)



[0, 20] by [0, 4000]

(a)

FIGURE 2.63 (Example 7)

So

$$2\pi r^2 + 2\pi r h = 1000$$

$$2\pi r^2 + 2\pi r\left(\frac{2000}{\pi r^2}\right) = 1000 \quad \text{Substitute } h = 2000/(\pi r^2).$$

$$2\pi r^2 + \frac{4000}{r} = 1000 \quad \text{Equation to be solved}$$

Solve Graphically

Figure 2.63 shows the graphs of $f(x) = 2\pi r^2 + 4000/r$ and $g(x) = 1000$. One point of intersection occurs when r is approximately 9.65. A second point of intersection occurs when r is approximately 4.62.

Because $h = 2000/(\pi r^2)$, the corresponding values for h are

$$h = \frac{2000}{\pi(4.619\ldots)^2} \approx 29.83 \text{ and } h = \frac{2000}{\pi(9.654\ldots)^2} \approx 6.83.$$

Interpret

With a surface area of 1000 cm^2, the cans either have a radius of 4.62 cm and a height of 29.83 cm or have a radius of 9.65 cm and a height of 6.83 cm.

Now try Exercise 37.

QUICK REVIEW 2.7 *(For help, go to Sections A.3. and P.5.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, find the missing numerator or denominator.

1. $\dfrac{2x}{x-3} = \dfrac{?}{x^2 + x - 12}$ **2.** $\dfrac{x-1}{x+1} = \dfrac{x^2-1}{?}$

In Exercises 3–6, find the LCD and rewrite the expression as a single fraction reduced to lowest terms.

3. $\dfrac{5}{12} + \dfrac{7}{18} - \dfrac{5}{6}$ **4.** $\dfrac{3}{x-1} - \dfrac{1}{x}$

5. $\dfrac{x}{2x+1} - \dfrac{2}{x-3}$

6. $\dfrac{x+1}{x^2-5x+6} - \dfrac{3x+11}{x^2-x-6}$

In Exercises 7–10, use the quadratic formula to find the zeros of the quadratic polynomials.

7. $2x^2 - 3x - 1$ **8.** $2x^2 - 5x - 1$

9. $3x^2 + 2x - 2$ **10.** $x^2 - 3x - 9$

SECTION 2.7 EXERCISES

In Exercises 1–6, solve the equation algebraically. Support your answer numerically and identify any extraneous solutions.

1. $\dfrac{x-2}{3} + \dfrac{x+5}{3} = \dfrac{1}{3}$ **2.** $x + 2 = \dfrac{15}{x}$

3. $x + 5 = \dfrac{14}{x}$ **4.** $\dfrac{1}{x} - \dfrac{2}{x-3} = 4$

5. $x + \dfrac{4x}{x-3} = \dfrac{12}{x-3}$ **6.** $\dfrac{3}{x-1} + \dfrac{2}{x} = 8$

In Exercises 7–12, solve the equation algebraically and graphically. Check for extraneous solutions.

7. $x + \dfrac{10}{x} = 7$ **8.** $x + 2 = \dfrac{15}{x}$

9. $x + \dfrac{12}{x} = 7$ **10.** $x + \dfrac{6}{x} = -7$

11. $2 - \dfrac{1}{x+1} = \dfrac{1}{x^2+x}$ **12.** $2 - \dfrac{3}{x+4} = \dfrac{12}{x^2+4x}$

In Exercises 13–18, solve the equation algebraically. Check for extraneous solutions. Support your answer graphically.

13. $\dfrac{3x}{x+5} + \dfrac{1}{x-2} = \dfrac{7}{x^2+3x-10}$

14. $\dfrac{4x}{x+4} + \dfrac{3}{x-1} = \dfrac{15}{x^2+3x-4}$

15. $\dfrac{x-3}{x} - \dfrac{3}{x+1} + \dfrac{3}{x^2+x} = 0$

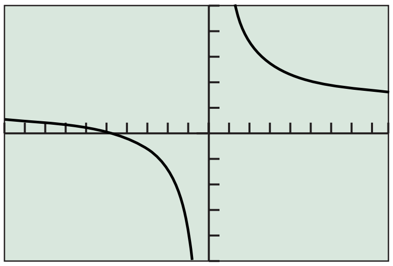
16. $\dfrac{x+2}{x} - \dfrac{4}{x-1} + \dfrac{2}{x^2-x} = 0$

17. $\dfrac{3}{x+2} + \dfrac{6}{x^2+2x} = \dfrac{3-x}{x}$

18. $\dfrac{x+3}{x} - \dfrac{2}{x+3} = \dfrac{6}{x^2+3x}$

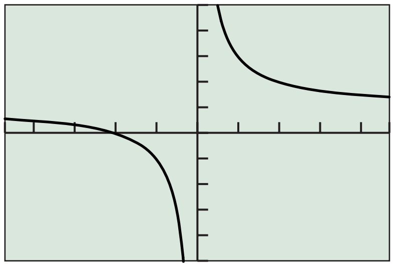
In Exercises 19–22, two possible solutions to the equation $f(x) = 0$ are listed. Use the given graph of $y = f(x)$ to decide which, if any, are extraneous.

19. $x = -5$ or $x = -2$



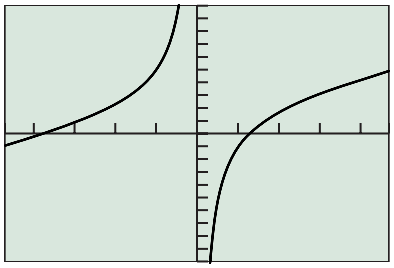
[–10, 8.8] by [–5, 5]

20. $x = -2$ or $x = 3$



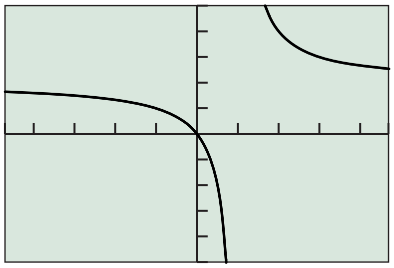
[–4.7, 4.7] by [–5, 5]

21. $x = -2$ or $x = 2$



[–4.7, 4.7] by [–10, 10]

22. $x = 0$ or $x = 3$



[–4.7, 4.7] by [–5, 5]

In Exercises 23–30, solve the equation.

23. $\dfrac{2}{x - 1} + x = 5$

24. $\dfrac{x^2 - 6x + 5}{x^2 - 2} = 3$

25. $\dfrac{x^2 - 2x + 1}{x + 5} = 0$

26. $\dfrac{3x}{x + 2} + \dfrac{2}{x - 1} = \dfrac{5}{x^2 + x - 2}$

27. $\dfrac{4x}{x + 4} + \dfrac{5}{x - 1} = \dfrac{15}{x^2 + 3x - 4}$

28. $\dfrac{3x}{x + 1} + \dfrac{5}{x - 2} = \dfrac{15}{x^2 - x - 2}$

29. $x^2 + \dfrac{5}{x} = 8$

30. $x^2 - \dfrac{3}{x} = 7$

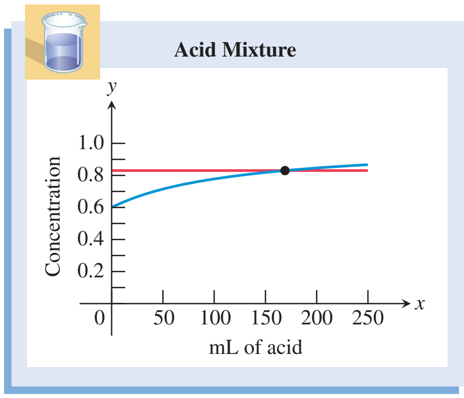
31. Acid Mixture Suppose that x mL of pure acid are added to 125 mL of a 60% acid solution. How many mL of pure acid must be added to obtain a solution of 83% acid?

(a) Explain why the concentration $C(x)$ of the new mixture is

$$C(x) = \frac{x + 0.6(125)}{x + 125}.$$

(b) Suppose the viewing window in the figure is used to find a solution to the problem. What is the equation of the horizontal line?

(c) **Writing to Learn** Write and solve an equation that answers the question of this problem. Explain your answer.



32. Acid Mixture Suppose that x mL of pure acid are added to 100 mL of a 35% acid solution.

(a) Express the concentration $C(x)$ of the new mixture as a function of x.

(b) Use a graph to determine how much pure acid should be added to the 35% solution to produce a new solution that is 75% acid.

(c) Solve (b) algebraically.

33. Breaking Even Mid Town Sports Apparel, Inc., has found that it needs to sell golf hats for \$2.75 each in order to be competitive. It costs \$2.12 to produce each hat, and it has weekly overhead costs of \$3000.

(a) Let x be the number of hats produced each week. Express the average cost (including overhead costs) of producing one hat as a function of x.

(b) Solve algebraically to find the number of golf hats that must be sold each week to make a profit. Support your answer graphically.

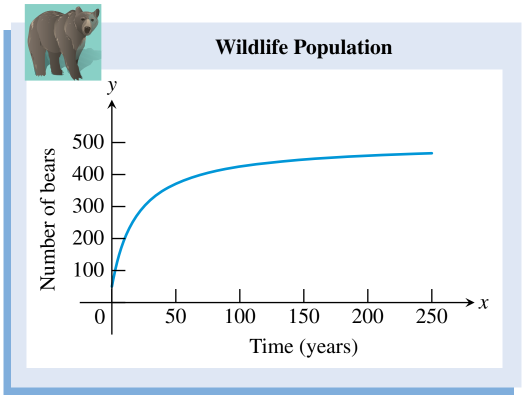
(c) **Writing to Learn** How many golf hats must be sold to make a profit of \$1000 in 1 week? Explain your answer.

34. Bear Population The number of bears at any time t (in years) in a federal game reserve is given by

$$P(t) = \frac{500 + 250t}{10 + 0.5t}.$$

(a) Find the population of bears when the value of t is 10, 40, and 100.

(b) Does the graph of the bear population have a horizontal asymptote? If so, what is it? If not, why not?



(c) Writing to Learn According to this model, what is the largest the bear population can become? Explain your answer.

35. Minimizing Perimeter Consider all rectangles with an area of 182 ft^2. Let x be the length of one side of such a rectangle.

(a) Express the perimeter P as a function of x.

(b) Find the dimensions of the rectangle that has the least perimeter. What is the least perimeter?

36. Group Activity Page Design Hendrix Publishing Co. wants to design a page that has a 0.75-in. left border, a 1.5-in. top border, and borders on the right and bottom of 1-in. They are to surround 40 $in.^2$ of print material. Let x be the width of the print material.

(a) Express the area of the page as a function of x.

(b) Find the dimensions of the page that has the least area. What is the least area?

37. Industrial Design Drake Cannery will pack peaches in 0.5-L cylindrical cans. Let x be the radius of the can in cm.

(a) Express the surface area S of the can as a function of x.

(b) Find the radius and height of the can if the surface area is 900 cm^2.

38. Group Activity Designing a Swimming Pool Thompson Recreation, Inc., wants to build a rectangular swimming pool with the top of the pool having surface area 1000 ft^2. The pool is required to have a walk of uniform width 2 ft surrounding it. Let x be the length of one side of the pool.

(a) Express the area of the plot of land needed for the pool and surrounding sidewalk as a function of x.

(b) Find the dimensions of the plot of land that has the least area. What is the least area?

39. Resistors The total electrical resistance R of two resistors connected in parallel with resistances R_1 and R_2 is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

One resistor has a resistance of 2.3 ohms. Let x be the resistance of the second resistor.

(a) Express the total resistance R as a function of x.

(b) Find the resistance of the second resistor if the total resistance of the pair is 1.7 ohms.

40. Designing Rectangles Consider all rectangles with an area of 200 m^2. Let x be the length of one side of such a rectangle.

(a) Express the perimeter P as a function of x.

(b) Find the dimensions of a rectangle whose perimeter is 70 m.

41. Swimming Pool Drainage Drains A and B are used to empty a swimming pool. Drain A alone can empty the pool in 4.75 h. Let t be the time it takes for drain B alone to empty the pool.

(a) Express as a function of t the part D of the drainage that can be done in 1 h with both drains open at the same time.

(b) Find graphically the time it takes for drain B alone to empty the pool if both drains, when open at the same time, can empty the pool in 2.6 h. Confirm algebraically.

42. Time-Rate Problem Josh rode his bike 17 mi from his home to Columbus, and then traveled 53 mi by car from Columbus to Dayton. Assume that the average rate of the car was 43 mph faster than the average rate of the bike.

(a) Express the total time required to complete the 70-mi trip (bike and car) as a function of the rate x of the bike.

(b) Find graphically the rate of the bike if the total time of the trip was 1 h 40 min. Confirm algebraically.

43. Late Expectations Table 2.20 shows the average number of remaining years to be lived by U.S. residents surviving to particular ages.

Table 2.20 Expectation for Remaining Life

Age (years)	Remaining Years
70	15.1
80	9.1
90	5.0
100	2.6

Source: National Vital Statistics Reports, Vol. 56, No. 9, December 2007.

(a) Draw a scatter plot of these data together with the model

$$E(a) = \frac{170}{a - 58},$$

where a is a person's age and E is the expected years remaining in the person's life.

(b) Use the model to predict how much longer the average U.S. 74-year-old will live.

44. Number of Wineries The number of wineries for several years is given in Table 2.21. Let $x = 0$ represent 1970, $x = 1$ represent 1971, and so forth. A model for these data is given by

$$y = 3000 - \frac{39{,}500}{x + 9}$$

(a) Graph the model together with a scatter plot of the data.

(b) Use the model to estimate the number of wineries in 2015.

Table 2.21 Number of Wineries

Year	Number
1980	912
1985	1375
1990	1625
1995	1813

Source: American Vintners Association as reported in USA TODAY on June 28, 2002.

Standardized Test Questions

45. True or False An extraneous solution of a rational equation is also a solution of the equation. Justify your answer.

46. True or False The equation $1/(x^2 - 4) = 0$ has no solution. Justify your answer.

In Exercises 47–50, solve the problem without using a calculator.

47. Multiple Choice Which of the following are the solutions of the equation $x - \frac{3x}{x + 2} = \frac{6}{x + 2}$?

(A) $x = -2$ or $x = 3$

(B) $x = -1$ or $x = 3$

(C) Only $x = -2$

(D) Only $x = 3$

(E) There are no solutions.

48. Multiple Choice Which of the following are the solutions of the equation $1 - \frac{3}{x} = \frac{6}{x^2 + 2x}$?

(A) $x = -2$ or $x = 4$

(B) $x = -3$ or $x = 0$

(C) $x = -3$ or $x = 4$

(D) Only $x = -3$

(E) There are no solutions.

49. Multiple Choice Which of the following are the solutions of the equation $\frac{x}{x + 2} + \frac{2}{x - 5} = \frac{14}{x^2 - 3x - 10}$?

(A) $x = -5$ or $x = 3$

(B) $x = -2$ or $x = 5$

(C) Only $x = 3$

(D) Only $x = -5$

(E) There are no solutions.

50. Multiple Choice Ten liters of a 20% acid solution are mixed with 30 liters of a 30% acid solution. Which of the following is the percent of acid in the final mixture?

(A) 21% **(B)** 22.5% **(C)** 25% **(D)** 27.5% **(E)** 28%

Explorations

51. Revisit Example 4 Consider the following equation, which we solved in Example 4.

$$\frac{x - 3}{x} + \frac{3}{x + 2} + \frac{6}{x^2 + 2x} = 0$$

Let $f(x) = \frac{x - 3}{x} + \frac{3}{x + 2} + \frac{6}{x^2 + 2x}$.

(a) Combine the fractions in $f(x)$ but do *not* reduce to lowest terms.

(b) What is the domain of f?

(c) Write f as a piecewise-defined function.

(d) Writing to Learn Graph f and explain how the graph supports your answers in (b) and (c).

Extending the Ideas

In Exercises 52–55, solve for x.

52. $y = 1 + \frac{1}{1 + x}$

53. $y = 1 - \frac{1}{1 - x}$

54. $y = 1 + \frac{1}{1 + \frac{1}{x}}$

55. $y = 1 + \frac{1}{1 + \frac{1}{1 - x}}$