2.6 Graphs of Rational Functions

What you'll learn about

- Rational Functions
- Transformations of the Reciprocal Function
- Limits and Asymptotes
- Analyzing Graphs of Rational Functions
- Exploring Relative Humidity

... and why

Rational functions are used in calculus and in scientific applications such as inverse proportions.

Rational Functions

Rational functions are ratios (or quotients) of polynomial functions.

DEFINITION Rational Functions

Let f and g be polynomial functions with $g(x) \neq 0$. Then the function given by

$$r(x) = \frac{f(x)}{g(x)}$$

is a **rational function**.

The domain of a rational function is the set of all real numbers except the zeros of its denominator. Every rational function is continuous on its domain.

EXAMPLE 1 Finding the Domain of a Rational Function

Find the domain of f and use limits to describe its behavior at value(s) of x not in its domain.

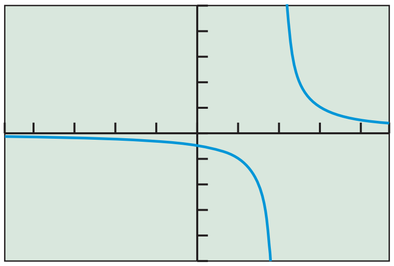
$$f(x) = \frac{1}{x - 2}$$

SOLUTION The domain of f is all real numbers $x \neq 2$. The graph in Figure 2.46 strongly suggests that f has a vertical asymptote at $x = 2$. As x approaches 2 from the left, the values of f decrease without bound. As x approaches 2 from the right, the values of f increase without bound. Using the notation introduced in Section 1.2 on page 92 we write

$$\lim_{x \to 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \infty.$$

The tables in Figure 2.47 support this visual evidence numerically.

Now try Exercise 1.



[–4.7, 4.7] by [–5, 5]

FIGURE 2.46 The graph of $f(x) = 1/(x - 2)$. (Example 1)

X	Y1
2	ERROR
2.01	100
2.02	50
2.03	33.333
2.04	25
2.05	20
2.06	16.667

Y1 = 1/(X–2)

(a)

X	Y1
2	ERROR
1.99	–100
1.98	–50
1.97	–33.33
1.96	–25
1.95	–20
1.94	–16.67

Y1 = 1/(X–2)

(b)

FIGURE 2.47 Table of values for $f(x) = 1/(x - 2)$ for values of x (a) to the right of 2, and (b) to the left of 2. (Example 1)

In Chapter 1 we defined horizontal and vertical asymptotes of the graph of a function $y = f(x)$. The line $y = b$ is a *horizontal asymptote* of the graph of f if

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b.$$

The line $x = a$ is a *vertical asymptote* of the graph of f if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

We can see from Figure 2.46 that $\lim_{x \to -\infty} 1/(x-2) = \lim_{x \to \infty} 1/(x-2) = 0$, so the line $y = 0$ is a horizontal asymptote of the graph of $f(x) = 1/(x-2)$. Because $\lim_{x \to 2^-} f(x) = -\infty$ and $\lim_{x \to 2^+} f(x) = \infty$, the line $x = 2$ is a vertical asymptote of $f(x) = 1/(x-2)$.

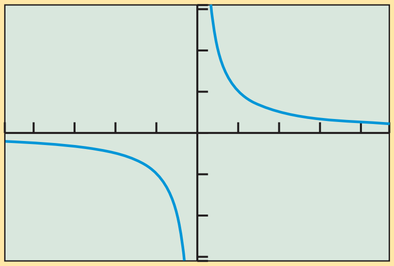
Transformations of the Reciprocal Function

One of the simplest rational functions is the reciprocal function

$$f(x) = \frac{1}{x},$$

which is one of the basic functions introduced in Chapter 1. It can be used to generate many other rational functions.

Here is what we know about the reciprocal function.



[–4.7, 4.7] by [–3.1, 3.1]

FIGURE 2.48 The graph of $f(x) = 1/x$.

BASIC FUNCTION The Reciprocal Function

$f(x) = \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Continuity: All $x \neq 0$

Decreasing on $(-\infty, 0)$ and $(0, \infty)$

Symmetric with respect to origin (an odd function)

Unbounded

No local extrema

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

End behavior: $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0$

EXPLORATION 1 Comparing Graphs of Rational Functions

1. Sketch the graph and find an equation for the function g whose graph is obtained from the reciprocal function $f(x) = 1/x$ by a translation of 2 units to the right.
2. Sketch the graph and find an equation for the function h whose graph is obtained from the reciprocal function $f(x) = 1/x$ by a translation of 5 units to the right, followed by a reflection across the x-axis.
3. Sketch the graph and find an equation for the function k whose graph is obtained from the reciprocal function $f(x) = 1/x$ by a translation of 4 units to the left, followed by a vertical stretch by a factor of 3, and finally a translation 2 units down.

The graph of any nonzero rational function of the form

$$g(x) = \frac{ax + b}{cx + d},\ c \neq 0$$

can be obtained through transformations of the graph of the reciprocal function. If the degree of the numerator is greater than or equal to the degree of the denominator, we can use polynomial division to rewrite the rational function.

EXAMPLE 2 Transforming the Reciprocal Function

Describe how the graph of the given function can be obtained by transforming the graph of the reciprocal function $f(x) = 1/x$. Identify the horizontal and vertical asymptotes and use limits to describe the corresponding behavior. Sketch the graph of the function.

(a) $g(x) = \dfrac{2}{x + 3}$ **(b)** $h(x) = \dfrac{3x - 7}{x - 2}$

SOLUTION

(a) $g(x) = \dfrac{2}{x + 3} = 2\left(\dfrac{1}{x + 3}\right) = 2f(x + 3)$

The graph of g is the graph of the reciprocal function shifted left 3 units and then stretched vertically by a factor of 2. So the lines $x = -3$ and $y = 0$ are vertical and horizontal asymptotes, respectively. Using limits we have $\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = 0$, $\lim_{x\to-3^+} g(x) = \infty$, and $\lim_{x\to-3^-} g(x) = -\infty$. The graph is shown in Figure 2.49a.

(b) We begin with polynomial division:

$$\begin{array}{r} 3 \\ x - 2\overline{)3x - 7} \\ \underline{3x - 6} \\ -1 \end{array}$$

So, $h(x) = \dfrac{3x - 7}{x - 2} = 3 - \dfrac{1}{x - 2} = -f(x - 2) + 3.$

Thus the graph of h is the graph of the reciprocal function translated 2 units to the right, followed by a reflection across the x-axis, and then translated 3 units up. (Note that the reflection must be executed before the vertical translation.) So the lines $x = 2$ and $y = 3$ are vertical and horizontal asymptotes, respectively. Using limits we have $\lim_{x\to\infty} h(x) = \lim_{x\to-\infty} h(x) = 3$, $\lim_{x\to2^+} g(x) = -\infty$, and $\lim_{x\to2^-} g(x) = \infty$. The graph is shown in Figure 2.49b. ***Now try Exercise 5.***

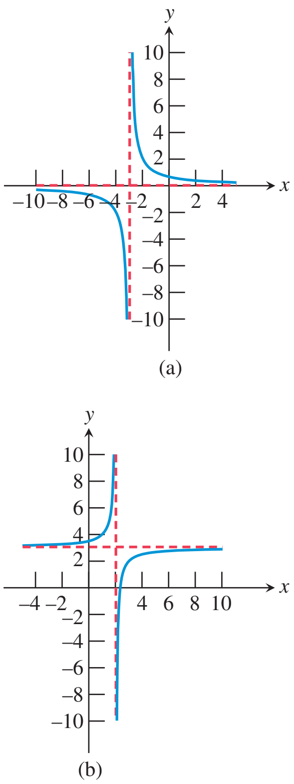


FIGURE 2.49 The graphs of (a) $g(x) = 2/(x + 3)$ and (b) $h(x) = (3x - 7)/(x - 2)$, with asymptotes shown in red.

Limits and Asymptotes

In Example 2 we found asymptotes by translating the known asymptotes of the reciprocal function. In Example 3, we use graphing and algebra to find an asymptote.

EXAMPLE 3 Finding Asymptotes

Find the horizontal and vertical asymptotes of $f(x) = (x^2 + 2)/(x^2 + 1)$. Use limits to describe the corresponding behavior of f.

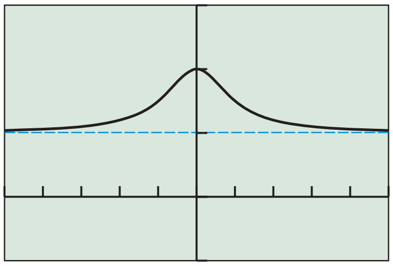
SOLUTION

Solve Graphically

The graph of f in Figure 2.50 suggests that

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1$$

and that there are no vertical asymptotes. The horizontal asymptote is $y = 1$.



[–5, 5] by [–1, 3]

FIGURE 2.50 The graph of $f(x) = (x^2 + 2)/(x^2 + 1)$ with its horizontal asymptote $y = 1$.

Solve Algebraically

Because the denominator $x^2 + 1 > 0$, the domain of f is all real numbers. So there are no vertical asymptotes. Using polynomial long division, we find that

$$f(x) = \frac{x^2 + 2}{x^2 + 1} = 1 + \frac{1}{x^2 + 1}.$$

When the value of $|x|$ is large, the denominator $x^2 + 1$ is a large positive number, and $1/(x^2 + 1)$ is a small positive number, getting closer to zero as $|x|$ increases. Therefore,

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 1,$$

so $y = 1$ is indeed a horizontal asymptote. ***Now try Exercise 19.***

Example 3 shows the connection between the end behavior of a rational function and its horizontal asymptote. We now generalize this relationship and summarize other features of the graph of a rational function:

Graph of a Rational Function

The graph of $y = f(x)/g(x) = (a_n x^n + \cdots)/(b_m x^m + \cdots)$ has the following characteristics:

1. **End behavior asymptote:**
   If $n < m$, the end behavior asymptote is the horizontal asymptote $y = 0$.
   If $n = m$, the end behavior asymptote is the horizontal asymptote $y = a_n/b_m$.
   If $n > m$, the end behavior asymptote is the quotient polynomial function $y = q(x)$, where $f(x) = g(x)q(x) + r(x)$. There is no horizontal asymptote.
2. **Vertical asymptotes:** These occur at the real zeros of the denominator, provided that the zeros are not also zeros of the numerator of equal or greater multiplicity.
3. ***x*-intercepts:** These occur at the real zeros of the numerator, which are not also zeros of the denominator.
4. ***y*-intercept:** This is the value of $f(0)$, if defined.

It is a good idea to find all of the asymptotes and intercepts when graphing a rational function. If the end behavior asymptote of a rational function is a slant line, we call it a **slant asymptote**, as illustrated in Example 4.

EXAMPLE 4 Graphing a Rational Function

Find the asymptotes and intercepts of the function $f(x) = x^3/(x^2 - 9)$ and graph the function.

SOLUTION The degree of the numerator is greater than the degree of the denominator, so the end behavior asymptote is the quotient polynomial. Using polynomial long division, we obtain

$$f(x) = \frac{x^3}{x^2 - 9} = x + \frac{9x}{x^2 - 9}.$$

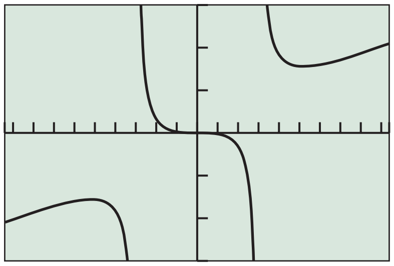
So the quotient polynomial is $q(x) = x$, a slant asymptote. Factoring the denominator,

$$x^2 - 9 = (x - 3)(x + 3),$$

(continued)

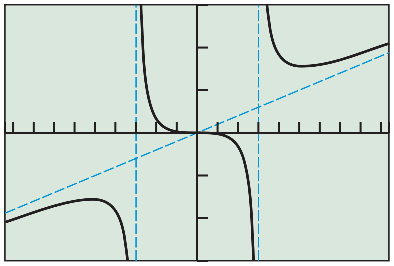
shows that the zeros of the denominator are $x = 3$ and $x = -3$. Consequently, $x = 3$ and $x = -3$ are the vertical asymptotes of f. The only zero of the numerator is 0, so $f(0) = 0$, and thus we see that the point (0, 0) is the only x-intercept and the y-intercept of the graph of f.

The graph of f in Figure 2.51a passes through (0, 0) and suggests the vertical asymptotes $x = 3$ and $x = -3$ and the slant asymptote $y = q(x) = x$. Figure 2.51b shows the graph of f with its asymptotes overlaid. *Now try Exercise 29.*



[–9.4, 9.4] by [–15, 15]

(a)



[–9.4, 9.4] by [–15, 15]

(b)

FIGURE 2.51 The graph of $f(x) = x^3/(x^2 - 9)$ (a) by itself and (b) with its asymptotes. (Example 4)

Analyzing Graphs of Rational Functions

Because the degree of the numerator of the rational function in Example 5 is less than the degree of the denominator, we know that the graph of the function has $y = 0$ as a horizontal asymptote.

EXAMPLE 5 Analyzing the Graph of a Rational Function

Find the intercepts and asymptotes, use limits to describe the behavior at the vertical asymptotes, and analyze and draw the graph of the rational function

$$f(x) = \frac{x - 1}{x^2 - x - 6}.$$

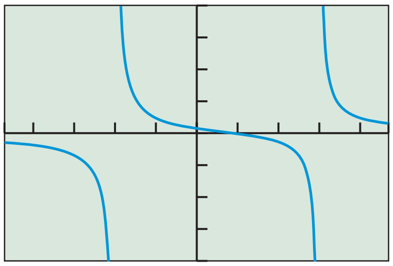
SOLUTION The numerator is zero when $x = 1$, so the x-intercept is 1. Because $f(0) = 1/6$, the y-intercept is 1/6. The denominator factors as

$$x^2 - x - 6 = (x - 3)(x + 2),$$

so there are vertical asymptotes at $x = -2$ and $x = 3$. From the comment preceding this example we know that the horizontal asymptote is $y = 0$. Figure 2.52 supports this information and allows us to conclude that

$$\lim_{x\to -2^-} f(x) = -\infty, \quad \lim_{x\to -2^+} f(x) = \infty, \quad \lim_{x\to 3^-} f(x) = -\infty, \text{ and } \lim_{x\to 3^+} f(x) = \infty.$$

Domain: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$
Range: All reals
Continuity: All $x \neq -2, 3$
Decreasing on $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$
Not symmetric
Unbounded
No local extrema
Horizontal asymptotes: $y = 0$
Vertical asymptotes: $x = -2$, $x = 3$
End behavior: $\lim_{x\to -\infty} f(x) = \lim_{x\to \infty} f(x) = 0$ *Now try Exercise 39.*



[–4.7, 4.7] by [–4, 4]

FIGURE 2.52 The graph of $f(x) = (x - 1)/(x^2 - x - 6)$. (Example 5)

The degrees of the numerator and denominator of the rational function in Example 6 are equal. Thus, we know that the graph of the function has $y = 2$, the quotient of the leading coefficients, as its end behavior asymptote.

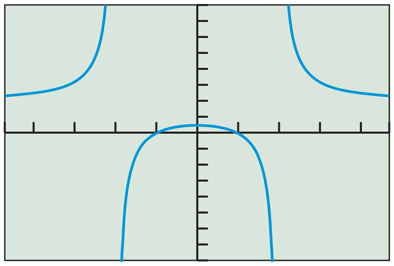
EXAMPLE 6 Analyzing the Graph of a Rational Function

Find the intercepts, analyze, and draw the graph of the rational function

$$f(x) = \frac{2x^2 - 2}{x^2 - 4}.$$

SOLUTION The numerator factors as

$$2x^2 - 2 = 2(x^2 - 1) = 2(x + 1)(x - 1),$$



[–4.7, 4.7] by [–8, 8]

FIGURE 2.53 The graph of $f(x) = (2x^2 - 2)/(x^2 - 4)$. It can be shown that f takes on no value between 1/2, the y-intercept, and 2, the horizontal asymptote. (Example 6)

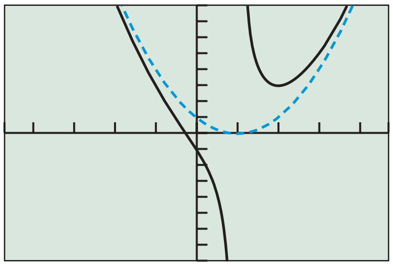
so the x-intercepts are -1 and 1. The y-intercept is $f(0) = 1/2$. The denominator factors as

$$x^2 - 4 = (x + 2)(x - 2),$$

so the vertical asymptotes are $x = -2$ and $x = 2$. From the comment preceding this example we know that $y = 2$ is the horizontal asymptote. Figure 2.53 supports this information and allows us to conclude that

$$\lim_{x \to -2^-} f(x) = \infty, \lim_{x \to -2^+} f(x) = -\infty, \lim_{x \to 2^-} f(x) = -\infty, \lim_{x \to 2^+} f(x) = \infty.$$

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
Range: $(-\infty, 1/2] \cup (2, \infty)$
Continuity: All $x \neq -2, 2$
Increasing on $(-\infty, -2)$ and $(-2, 0]$; decreasing on $[0, 2)$ and $(2, \infty)$
Symmetric with respect to the y-axis (an even function)
Unbounded
Local maximum of 1/2 at $x = 0$
Horizontal asymptotes: $y = 2$
Vertical asymptotes: $x = -2, x = 2$
End behavior: $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 2$ *Now try Exercise 41.*



[–4.7, 4.7] by [–8, 8]

FIGURE 2.54 The graph of $f(x) = (x^3 - 3x^2 + 3x + 1)/(x - 1)$ as a solid black line and its end behavior asymptote $y = x^2 - 2x + 1$ as a dashed blue line. (Examples 7 and 8)

In Examples 7 and 8 we will investigate the rational function

$$f(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1}.$$

The degree of the numerator of f exceeds the degree of the denominator by 2. Thus, there is no horizontal asymptote. We will see that the end behavior asymptote is a polynomial of degree 2.

EXAMPLE 7 Finding an End Behavior Asymptote

Find the end behavior asymptote of

$$f(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1}$$

and graph it together with f in two windows:

(a) one showing the details around the vertical asymptote of f,

(b) one showing a graph of f that resembles its end behavior asymptote.

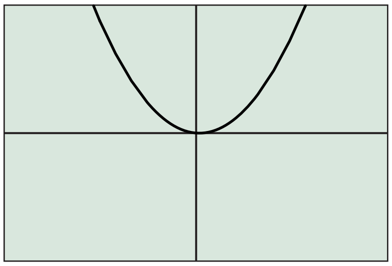
SOLUTION The graph of f has a vertical asymptote at $x = 1$. Divide $x^3 - 3x^2 + 3x + 1$ by $x - 1$ to show that

$$f(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1} = x^2 - 2x + 1 + \frac{2}{x - 1}.$$

The end behavior asymptote of f is $y = x^2 - 2x + 1$.

(a) The graph of f in Figure 2.54 shows the details around the vertical asymptote. We have also overlaid the graph of its end behavior asymptote as a dashed line.

(b) If we draw the graph of $f(x) = (x^3 - 3x^2 + 3x + 1)/(x - 1)$ and its end behavior asymptote $y = x^2 - 2x + 1$ in a large enough viewing window, the two graphs will appear to be identical (Figure 2.55). *Now try Exercise 47.*



[–40, 40] by [–500, 500]

FIGURE 2.55 The graphs of $f(x) = (x^3 - 3x^2 + 3x + 1)/(x - 1)$ and its end behavior asymptote $y = x^2 - 2x + 1$ appear to be identical. (Example 7)

EXAMPLE 8 Analyzing the Graph of a Rational Function

Find the intercepts, analyze, and draw the graph of the rational function

$$f(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1}.$$

SOLUTION f has only one x-intercept and we can use the graph of f in Figure 2.54 to show that it is about -0.26. The y-intercept is $f(0) = -1$. The vertical asymptote is $x = 1$ as we have seen. We know that the graph of f does not have a horizontal asymptote, and from Example 7 we know that the end behavior asymptote is $y = x^2 - 2x + 1$. We can also use Figure 2.54 to show that f has a local minimum of 3 at $x = 2$. Figure 2.54 supports this information and allows us to conclude that

$$\lim_{x\to 1^-} f(x) = -\infty \text{ and } \lim_{x\to 1^+} f(x) = \infty.$$

Domain: All $x \neq 1$
Range: All reals
Continuity: All $x \neq 1$
Decreasing on $(-\infty, 1)$ and $(1, 2]$; increasing on $[2, \infty)$
Not symmetric
Unbounded
Local minimum of 3 at $x = 2$
No horizontal asymptotes; end behavior asymptote: $y = x^2 - 2x + 1$
Vertical asymptote: $x = 1$
End behavior: $\lim_{x\to -\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$

Now try Exercise 55.

Exploring Relative Humidity

The phrase *relative humidity* is familiar from everyday weather reports. **Relative humidity** is the ratio of constant vapor pressure to saturated vapor pressure. So, relative humidity is inversely proportional to saturated vapor pressure.

Chapter Opener Problem (from page 157)

Problem: Determine the relative humidity values that correspond to the saturated vapor pressures of 12, 24, 36, 48, and 60 millibars, at a constant vapor pressure of 12 millibars. (In practice, saturated vapor pressure increases as the temperature increases.)

Solution: Relative humidity (RH) is found by dividing constant vapor pressure (CVP) by saturated vapor pressure (SVP). So, for example, for SVP = 24 millibars and CVP = 12 millibars, RH = 12/24 = 1/2 = 0.5 = 50%. See the table below, which is based on CVP = 12 millibars with increasing temperature.

SVP (millibars)	RH (%)
12	100
24	50
36	$33.\overline{3}$
48	25
60	20

QUICK REVIEW 2.6 *(For help, go to Section 2.4.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–6, use factoring to find the real zeros of the function.

1. $f(x) = 2x^2 + 5x - 3$ **2.** $f(x) = 3x^2 - 2x - 8$

3. $g(x) = x^2 - 4$ **4.** $g(x) = x^2 - 1$

5. $h(x) = x^3 - 1$ **6.** $h(x) = x^2 + 1$

In Exercises 7–10, find the quotient and remainder when $f(x)$ is divided by $d(x)$.

7. $f(x) = 2x + 1, d(x) = x - 3$

8. $f(x) = 4x + 3, d(x) = 2x - 1$

9. $f(x) = 3x - 5, d(x) = x$

10. $f(x) = 5x - 1, d(x) = 2x$

SECTION 2.6 EXERCISES

In Exercises 1–4, find the domain of the function f. Use limits to describe the behavior of f at value(s) of x not in its domain.

1. $f(x) = \dfrac{1}{x + 3}$ **2.** $f(x) = \dfrac{-3}{x - 1}$

3. $f(x) = \dfrac{-1}{x^2 - 4}$ **4.** $f(x) = \dfrac{2}{x^2 - 1}$

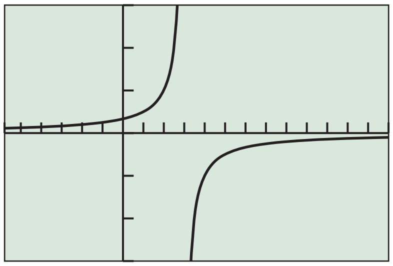
In Exercises 5–10, describe how the graph of the given function can be obtained by transforming the graph of the reciprocal function $g(x) = 1/x$. Identify the horizontal and vertical asymptotes and use limits to describe the corresponding behavior. Sketch the graph of the function.

5. $f(x) = \dfrac{1}{x - 3}$ **6.** $f(x) = -\dfrac{2}{x + 5}$

7. $f(x) = \dfrac{2x - 1}{x + 3}$ **8.** $f(x) = \dfrac{3x - 2}{x - 1}$

9. $f(x) = \dfrac{5 - 2x}{x + 4}$ **10.** $f(x) = \dfrac{4 - 3x}{x - 5}$

In Exercises 11–14, evaluate the limit based on the graph of f shown.

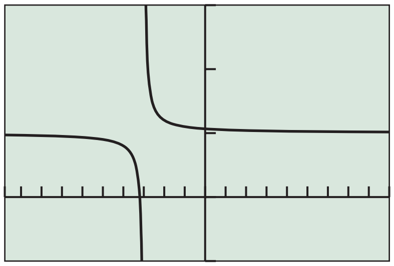


[–5.8, 13] by [–3, 3]

11. $\lim_{x \to 3^-} f(x)$ **12.** $\lim_{x \to 3^+} f(x)$

13. $\lim_{x \to \infty} f(x)$ **14.** $\lim_{x \to -\infty} f(x)$

In Exercises 15–18, evaluate the limit based on the graph of f shown.



[–9.8, 9] by [–5, 15]

15. $\lim_{x \to -3^+} f(x)$ **16.** $\lim_{x \to -3^-} f(x)$

17. $\lim_{x \to -\infty} f(x)$ **18.** $\lim_{x \to \infty} f(x)$

In Exercises 19–22, find the horizontal and vertical asymptotes of $f(x)$. Use limits to describe the corresponding behavior.

19. $f(x) = \dfrac{2x^2 - 1}{x^2 + 3}$ **20.** $f(x) = \dfrac{3x^2}{x^2 + 1}$

21. $f(x) = \dfrac{2x + 1}{x^2 - x}$ **22.** $f(x) = \dfrac{x - 3}{x^2 + 3x}$

In Exercises 23–30, find the asymptotes and intercepts of the function, and graph the function.

23. $g(x) = \dfrac{x - 2}{x^2 - 2x - 3}$ **24.** $g(x) = \dfrac{x + 2}{x^2 + 2x - 3}$

25. $h(x) = \dfrac{2}{x^3 - x}$ **26.** $h(x) = \dfrac{3}{x^3 - 4x}$

27. $f(x) = \dfrac{2x^2 + x - 2}{x^2 - 1}$ **28.** $g(x) = \dfrac{-3x^2 + x + 12}{x^2 - 4}$

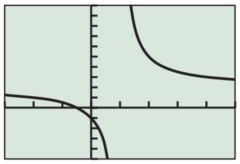
29. $f(x) = \dfrac{x^2 - 2x + 3}{x + 2}$ **30.** $g(x) = \dfrac{x^2 - 3x - 7}{x + 3}$

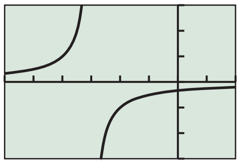
In Exercises 31–36, match the rational function with its graph. Identify the viewing window and the scale used on each axis.

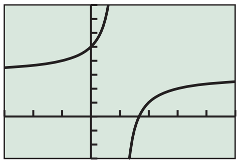
31. $f(x) = \dfrac{1}{x - 4}$ **32.** $f(x) = -\dfrac{1}{x + 3}$

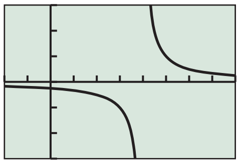
33. $f(x) = 2 + \dfrac{3}{x - 1}$ **34.** $f(x) = 1 + \dfrac{1}{x + 3}$

35. $f(x) = -1 + \dfrac{1}{4 - x}$ **36.** $f(x) = 3 - \dfrac{2}{x - 1}$


(a)


(b)


(c)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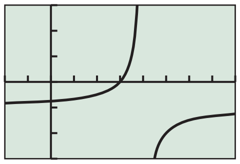
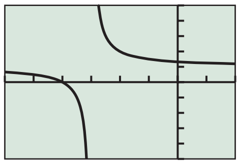

(e)


(f)

In Exercises 37–44, find the intercepts and asymptotes, use limits to describe the behavior at the vertical asymptotes, and analyze and draw the graph of the given rational function.

37. $f(x) = \dfrac{2}{2x^2 - x - 3}$ **38.** $g(x) = \dfrac{2}{x^2 + 4x + 3}$

39. $h(x) = \dfrac{x - 1}{x^2 - x - 12}$ **40.** $k(x) = \dfrac{x + 1}{x^2 - 3x - 10}$

41. $f(x) = \dfrac{x^2 + x - 2}{x^2 - 9}$ **42.** $g(x) = \dfrac{x^2 - x - 2}{x^2 - 2x - 8}$

43. $h(x) = \dfrac{x^2 + 2x - 3}{x + 2}$ **44.** $k(x) = \dfrac{x^2 - x - 2}{x - 3}$

In Exercises 45–50, find the end behavior asymptote of the given rational function f and graph it together with f in two windows:

(a) One showing the details around the vertical asymptote(s) of f.
(b) One showing a graph of f that resembles its end behavior asymptote.

45. $f(x) = \dfrac{x^2 - 2x + 3}{x - 5}$ **46.** $f(x) = \dfrac{2x^2 + 2x - 3}{x + 3}$

47. $f(x) = \dfrac{x^3 - x^2 + 1}{x + 2}$ **48.** $f(x) = \dfrac{x^3 + 1}{x - 1}$

49. $f(x) = \dfrac{x^4 - 2x + 1}{x - 2}$ **50.** $f(x) = \dfrac{x^5 + 1}{x^2 + 1}$

In Exercises 51–56, find the intercepts, analyze, and graph the given rational function.

51. $f(x) = \dfrac{3x^2 - 2x + 4}{x^2 - 4x + 5}$ **52.** $g(x) = \dfrac{4x^2 + 2x}{x^2 - 4x + 8}$

53. $h(x) = \dfrac{x^3 - 1}{x - 2}$ **54.** $k(x) = \dfrac{x^3 - 2}{x + 2}$

55. $f(x) = \dfrac{x^3 - 2x^2 + x - 1}{2x - 1}$ **56.** $g(x) = \dfrac{2x^3 - 2x^2 - x + 5}{x - 2}$

In Exercises 57–62, find the intercepts, vertical asymptotes, and end behavior asymptote, and graph the function together with its end behavior asymptote.

57. $h(x) = \dfrac{x^4 + 1}{x + 1}$ **58.** $k(x) = \dfrac{2x^5 + x^2 - x + 1}{x^2 - 1}$

59. $f(x) = \dfrac{x^5 - 1}{x + 2}$ **60.** $g(x) = \dfrac{x^5 + 1}{x - 1}$

61. $h(x) = \dfrac{2x^3 - 3x + 2}{x^3 - 1}$ **62.** $k(x) = \dfrac{3x^3 + x - 4}{x^3 + 1}$

Standardized Test Questions

63. True or False A rational function must have a vertical asymptote. Justify your answer.

64. True or False $f(x) = \dfrac{x^2 - x}{\sqrt{x^2 + 4}}$ is a rational function. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve the problem.

65. Multiple Choice Let $f(x) = \dfrac{-2}{x^2 + 3x}$. What values of x have to be excluded from the domain of f?

(A) Only 0 **(B)** Only 3 **(C)** Only -3
(D) Only 0, 3 **(E)** Only 0, -3

66. Multiple Choice Let $g(x) = \dfrac{2}{x + 3}$. Which of the transformations of $f(x) = \dfrac{2}{x}$ produce the graph of g?

(A) Translate the graph of f left 3 units.
(B) Translate the graph of f right 3 units.
(C) Translate the graph of f down 3 units.
(D) Translate the graph of f up 3 units.
(E) Vertically stretch the graph of f by a factor of 2.

67. Multiple Choice Let $f(x) = \dfrac{x^2}{x + 5}$. Which of the following statements is true about the graph of f?

(A) There is no vertical asymptote.
(B) There is a horizontal asymptote but no vertical asymptote.
(C) There is a slant asymptote but no vertical asymptote.
(D) There is a vertical asymptote and a slant asymptote.
(E) There is a vertical and horizontal asymptote.

68. Multiple Choice What is the degree of the end behavior asymptote of $f(x) = \dfrac{x^8 + 1}{x^4 + 1}$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Explorations

69. Group Activity *Work in groups of two.* Compare the functions $f(x) = \dfrac{x^2 - 9}{x - 3}$ and $g(x) = x + 3$.

(a) Are the domains equal?

(b) Does f have a vertical asymptote? Explain.

(c) Explain why the graphs appear to be identical.

(d) Are the functions identical?

70. Group Activity Explain why the functions are identical or not. Include the graphs and a comparison of the functions' asymptotes, intercepts, and domain.

(a) $f(x) = \dfrac{x^2 + x - 2}{x - 1}$ and $g(x) = x + 2$

(b) $f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$

(c) $f(x) = \dfrac{x^2 - 1}{x^3 - x^2 - x + 1}$ and $g(x) = \dfrac{1}{x - 1}$

(d) $f(x) = \dfrac{x - 1}{x^2 + x - 2}$ and $g(x) = \dfrac{1}{x + 2}$

71. Boyle's Law This ideal gas law states that the volume of an enclosed gas at a fixed temperature varies inversely as the pressure.

(a) Writing to Learn Explain why Boyle's Law yields both a rational function model and a power function model.

(b) Which power functions are also rational functions?

(c) If the pressure of a 2.59-L sample of nitrogen gas at a temperature of 291 K is 0.866 atm, what would the volume be at a pressure of 0.532 atm if the temperature does not change?

72. Light Intensity Aileen and Malachy gathered the data in Table 2.19 using a 75-watt lightbulb and a Calculator-Based Laboratory™ (CBL™) with a light-intensity probe.

(a) Draw a scatter plot of the data in Table 2.19.

(b) Find an equation for the data assuming it has the form $f(x) = k/x^2$ for some constant k. Explain your method for choosing k.

(c) Superimpose the regression curve on the scatter plot.

(d) Use the regression model to predict the light intensity at distances of 2.2 m and 4.4 m.

Table 2.19 Light-Intensity Data for a 75-W Lightbulb

Distance (m)	Intensity (W/m^2)
1.0	6.09
1.5	2.51
2.0	1.56
2.5	1.08
3.0	0.74

Extending the Ideas

In Exercises 73–76, graph the function. Express the function as a piecewise-defined function without absolute value, and use the result to confirm the graph's asymptotes and intercepts algebraically.

73. $h(x) = \dfrac{2x - 3}{|x| + 2}$

74. $h(x) = \dfrac{3x + 5}{|x| + 3}$

75. $f(x) = \dfrac{5 - 3x}{|x| + 4}$

76. $f(x) = \dfrac{2 - 2x}{|x| + 1}$

77. Describe how the graph of a nonzero rational function

$$f(x) = \frac{ax + b}{cx + d},\ c \neq 0$$

can be obtained from the graph of $y = 1/x$. (*Hint:* Use long division.)

78. Writing to Learn Let $f(x) = 1 + 1/(x - 1/x)$ and $g(x) = (x^3 + x^2 - x)/(x^3 - x)$. Does $f = g$? Support your answer by making a comparative analysis of all of the features of f and g, including asymptotes, intercepts, and domain.