2.4 Real Zeros of Polynomial Functions

What you'll learn about

- Long Division and the Division Algorithm
- Remainder and Factor Theorems
- Synthetic Division
- Rational Zeros Theorem
- Upper and Lower Bounds

... and why

These topics help identify and locate the real zeros of polynomial functions.

Long Division and the Division Algorithm

We have seen that factoring a polynomial reveals its zeros and much about its graph. Polynomial division gives us new and better ways to factor polynomials. First we observe that the division of polynomials closely resembles the division of integers:

112	$1x^2 + 1x + 2$	← Quotient
$32\overline{)3587}$	$3x + 2\overline{)3x^3 + 5x^2 + 8x + 7}$	← Dividend
32	$3x^3 + 2x^2$	← Multiply: $1x^2 \cdot (3x + 2)$
387	$3x^2 + 8x + 7$	← Subtract
32	$3x^2 + 2x$	← Multiply: $1x \cdot (3x + 2)$
67	$6x + 7$	← Subtract
64	$6x + 4$	← Multiply: $2 \cdot (3x + 2)$
3	3	← Remainder

Division, whether integer or polynomial, involves a *dividend* divided by a *divisor* to obtain a *quotient* and a *remainder*. We can check and summarize our result with an equation of the form

$$(\text{Divisor})(\text{Quotient}) + \text{Remainder} = \text{Dividend}.$$

For instance, to check or summarize the long divisions shown above we could write

$$32 \times 112 + 3 = 3587 \qquad (3x + 2)(x^2 + x + 2) + 3 = 3x^3 + 5x^2 + 8x + 7.$$

The *division algorithm* contains such a summary *polynomial equation*, but with the dividend written on the left side of the equation.

Division Algorithm for Polynomials

Let $f(x)$ and $d(x)$ be polynomials with the degree of f greater than or equal to the degree of d, and $d(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$, called the **quotient** and **remainder**, such that

$$f(x) = d(x) \cdot q(x) + r(x), \tag{1}$$

where either $r(x) = 0$ or the degree of r is less than the degree of d.

The function $f(x)$ in the division algorithm is the **dividend**, and $d(x)$ is the **divisor**. If $r(x) = 0$, we say $d(x)$ **divides evenly** into $f(x)$.

The summary statement (1) is sometimes written in *fraction form* as follows:

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \tag{2}$$

For instance, to summarize the polynomial division example above we could write

$$\frac{3x^3 + 5x^2 + 8x + 7}{3x + 2} = x^2 + x + 2 + \frac{3}{3x + 2}.$$

EXAMPLE 1 Using Polynomial Long Division

Use long division to find the quotient and remainder when $2x^4 - x^3 - 2$ is divided by $2x^2 + x + 1$. Write a summary statement in both polynomial and fraction form.

(continued)

SOLUTION

Solve Algebraically

$$\begin{array}{r} x^2 - x \quad \leftarrow \text{Quotient} \\ 2x^2 + x + 1 \overline{)\, 2x^4 - x^3 + 0x^2 + 0x - 2} \\ \underline{2x^4 + x^3 + x^2} \\ -2x^3 - x^2 + 0x - 2 \\ \underline{-2x^3 - x^2 - x} \\ x - 2 \quad \leftarrow \text{Remainder} \end{array}$$

The division algorithm yields the polynomial form

$$2x^4 - x^3 - 2 = (2x^2 + x + 1)(x^2 - x) + (x - 2).$$

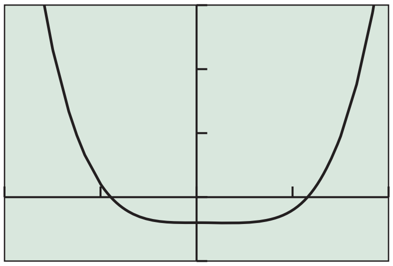
Using equation 2, we obtain the fraction form

$$\frac{2x^4 - x^3 - 2}{2x^2 + x + 1} = x^2 - x + \frac{x - 2}{2x^2 + x + 1}.$$

Support Graphically

Figure 2.34 supports the polynomial form of the summary statement.

Now try Exercise 1.



[–2, 2] by [–5, 15]

FIGURE 2.34 The graphs of $y_1 = 2x^4 - x^3 - 2$ and $y_2 = (2x^2 + x + 1)(x^2 - x) + (x - 2)$ are a perfect match. (Example 1)

Remainder and Factor Theorems

An important special case of the division algorithm occurs when the divisor is of the form $d(x) = x - k$, where k is a real number. Because the degree of $d(x) = x - k$ is 1, the remainder is a real number. We obtain the following simplified summary statement for the division algorithm:

$$f(x) = (x - k)q(x) + r \tag{3}$$

We use this special case of the division algorithm throughout the rest of the section.

Using equation (3), we evaluate the polynomial $f(x)$ at $x = k$:

$$f(k) = (k - k)q(k) + r = 0 \cdot q(k) + r = 0 + r = r$$

So $f(k) = r$, which is the remainder. This reasoning yields the following theorem.

THEOREM Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Example 2 shows a clever use of the Remainder Theorem that gives information about the factors, zeros, and x-intercepts.

EXAMPLE 2 Using the Remainder Theorem

Find the remainder when $f(x) = 3x^2 + 7x - 20$ is divided by

(a) $x - 2$ **(b)** $x + 1$ **(c)** $x + 4$.

SOLUTION

Solve Numerically (by hand)

(a) We can find the remainder without doing long division! Using the Remainder Theorem with $k = 2$ we find that

$$r = f(2) = 3(2)^2 + 7(2) - 20 = 12 + 14 - 20 = 6.$$

X	Y_1	
−4	0	
−3	−14	
−2	−22	
−1	−24	
0	−20	
1	−10	
2	6	

$Y_1 = 3X\text{^}2+7X-20$

FIGURE 2.35 Table for $f(x) = 3x^2 + 7x - 20$ showing the remainders obtained when $f(x)$ is divided by $x - k$, for $k = -4, -3, \ldots, 1, 2$.

Proof of the Factor Theorem

If $f(x)$ has a factor $x - k$, there is a polynomial $g(x)$ such that

$$f(x) = (x - k)g(x) = (x - k)g(x) + 0.$$

By the uniqueness condition of the division algorithm, $g(x)$ is the quotient and 0 is the remainder, and by the Remainder Theorem, $f(k) = 0$.

Conversely, if $f(k) = 0$, the remainder $r = 0$, $x - k$ divides evenly into $f(x)$, and $x - k$ is a factor of $f(x)$.

(b) $r = f(-1) = 3(-1)^2 + 7(-1) - 20 = 3 - 7 - 20 = -24.$

(c) $r = f(-4) = 3(-4)^2 + 7(-4) - 20 = 48 - 28 - 20 = 0.$

Interpret

Because the remainder in part (c) is 0, $x + 4$ *divides evenly* into $f(x) = 3x^2 + 7x - 20$. So, $x + 4$ is a factor of $f(x) = 3x^2 + 7x - 20$, -4 is a solution of $3x^2 + 7x - 20 = 0$, and -4 is an x-intercept of the graph of $y = 3x^2 + 7x - 20$. We know all of this without ever dividing, factoring, or graphing!

Support Numerically (using a grapher)

We can find the remainders of several division problems at once using the table feature of a grapher (Figure 2.35). ***Now try Exercise 13.***

Our interpretation of Example 2c leads us to the following theorem.

THEOREM Factor Theorem

A polynomial function $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Applying the ideas of the Factor Theorem to Example 2, we can factor $f(x) = 3x^2 + 7x - 20$ by dividing it by the known factor $x + 4$.

$$\begin{array}{r}
3x - 5 \\
x + 4\overline{)\,3x^2 + 7x - 20} \\
\underline{3x^2 + 12x} \\
-5x - 20 \\
\underline{-5x - 20} \\
0
\end{array}$$

So, $f(x) = 3x^2 + 7x - 20 = (x + 4)(3x - 5)$. In this case, there really is no need to use long division or fancy theorems; traditional factoring methods can do the job. However, for polynomials of degree 3 and higher, these sophisticated methods can be quite helpful in solving equations and finding factors, zeros, and x-intercepts. Indeed, the Factor Theorem ties in nicely with earlier connections we have made in the following way.

Fundamental Connections for Polynomial Functions

For a polynomial function f and a real number k, the following statements are equivalent:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$.
2. k is a zero of the function f.
3. k is an x-intercept of the graph of $y = f(x)$.
4. $x - k$ is a factor of $f(x)$.

Synthetic Division

We continue with the important special case of polynomial division with the divisor $x - k$. The Remainder Theorem gave us a way to find remainders in this case without long division. We now learn a method for finding both quotients and remainders for division by $x - k$ without long division. This shortcut method for the division of a polynomial by a linear divisor $x - k$ is **synthetic division**.

We illustrate the evolution of this method below, progressing from long division through two intermediate stages to synthetic division.

Moving from stage to stage, focus on the coefficients and their relative positions. Moving from stage 1 to stage 2, we suppress the variable x and the powers of x, and then from stage 2 to stage 3, we eliminate unneeded duplications and collapse vertically.

Stage 1
Long Division

$$
\begin{array}{r}
2x^2 + 3x + 4 \\
x - 3\,\overline{)\,2x^3 - 3x^2 - 5x - 12} \\
\underline{2x^3 - 6x^2} \\
3x^2 - 5x - 12 \\
\underline{3x^2 - 9x} \\
4x - 12 \\
\underline{4x - 12} \\
0
\end{array}
$$

Stage 2
Variables Suppressed

$$
\begin{array}{rrrrr}
 & 2 & 3 & 4 & \\
-3\,) & 2 & -3 & -5 & -12 \\
 & \underline{2} & \underline{-6} & & \\
 & & 3 & -5 & -12 \\
 & & \underline{3} & \underline{-9} & \\
 & & & 4 & -12 \\
 & & & \underline{4} & \underline{-12} \\
 & & & & 0
\end{array}
$$

Stage 3
Collapsed Vertically

$$
\begin{array}{r|rrrrl}
-3 & 2 & -3 & -5 & -12 & \text{Dividend} \\
 & & -6 & -9 & -12 & \\
\hline
 & 2 & 3 & 4 & 0 & \text{Quotient, remainder}
\end{array}
$$

Finally, from stage 3 to stage 4, we change the sign of the number representing the divisor and the signs of the numbers on the second line of our division scheme. These sign changes yield two advantages:

- The number standing for the divisor $x - k$ is now k, its zero.
- Changing the signs in the second line allows us to add rather than subtract.

Stage 4
Synthetic Division

$$
\begin{array}{lr|rrrrl}
\text{Zero of divisor} \rightarrow & 3 & 2 & -3 & -5 & -12 & \text{Dividend} \\
 & & & 6 & 9 & 12 & \\
\hline
 & & 2 & 3 & 4 & 0 & \text{Quotient, remainder}
\end{array}
$$

With stage 4 we have achieved our goal of synthetic division, a highly streamlined version of dividing a polynomial by $x - k$. How does this "bare bones" division work? Example 3 explains the steps.

EXAMPLE 3 Using Synthetic Division

Divide $2x^3 - 3x^2 - 5x - 12$ by $x - 3$ using synthetic division and write a summary statement in fraction form.

SOLUTION

Set Up

The zero of the divisor $x - 3$ is 3, which we put in the divisor position. Because the dividend is in standard form, we write its coefficients in order in the dividend position, *making sure to use a zero as a placeholder for any missing term*. We leave space for the line for products and draw a horizontal line below the space. (See below.)

Calculate

- Because the leading coefficient of the dividend must be the leading coefficient of the quotient, copy the 2 into the first quotient position.

$$
\begin{array}{lr|rrrrl}
\textbf{Zero of Divisor} & 3 & 2 & -3 & -5 & -12 & \textbf{Dividend} \\
\text{Line for products} & & & & & & \\
\hline
 & & 2 & & & &
\end{array}
$$

- Multiply the zero of the divisor (3) by the most recently determined coefficient of the quotient (2). Write the product above the line and one column to the right.

- Add the next coefficient of the dividend to the product just found and record the sum below the line in the same column.
- Repeat the "multiply" and "add" steps until the last row is completed.

Zero of Divisor	3 |	2	−3	−5	−12	**Dividend**
Line for products			6	9	12	
Line for sums		2	3	4	0	**Remainder**
		Quotient				

Interpret

The numbers in the last line are the coefficients of the quotient polynomial and the remainder. The quotient must be a quadratic function. (Why?) So the quotient is $2x^2 + 3x + 4$ and the remainder is 0. We conclude that

$$\frac{2x^3 - 3x^2 - 5x - 12}{x - 3} = 2x^2 + 3x + 4, \; x \neq 3.$$

Now try Exercise 7.

Rational Zeros Theorem

Real zeros of polynomial functions are either **rational zeros**—zeros that are rational numbers—or **irrational zeros**—zeros that are irrational numbers. For example,

$$f(x) = 4x^2 - 9 = (2x + 3)(2x - 3)$$

has the rational zeros $-3/2$ and $3/2$, and

$$f(x) = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$$

has the irrational zeros $-\sqrt{2}$ and $\sqrt{2}$.

The Rational Zeros Theorem tells us how to make a list of all potential rational zeros for a polynomial function with integer coefficients.

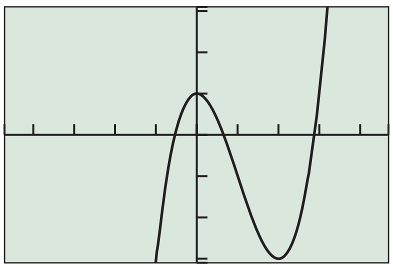
THEOREM Rational Zeros Theorem

Suppose f is a polynomial function of degree $n \geq 1$ of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

with every coefficient an integer and $a_0 \neq 0$. If $x = p/q$ is a rational zero of f, where p and q have no common integer factors other than ± 1, then

- p is an integer factor of the constant coefficient a_0, and
- q is an integer factor of the leading coefficient a_n.



[–4.7, 4.7] by [–3.1, 3.1]

FIGURE 2.36 The function $f(x) = x^3 - 3x^2 + 1$ has three real zeros. (Example 4)

EXAMPLE 4 Finding the Rational Zeros

Find the rational zeros of $f(x) = x^3 - 3x^2 + 1$.

SOLUTION Because the leading and constant coefficients are both 1, according to the Rational Zeros Theorem, the only potential rational zeros of f are 1 and -1. We check to see whether they are in fact zeros of f:

$$f(1) = (1)^3 - 3(1)^2 + 1 = -1 \neq 0$$
$$f(-1) = (-1)^3 - 3(-1)^2 + 1 = -3 \neq 0$$

So f has no rational zeros. Figure 2.36 shows that the graph of f has three x-intercepts. Therefore, f has three real zeros. All three must be irrational numbers.

Now try Exercise 33.

In Example 4 the Rational Zeros Theorem gave us only two candidates for rational zeros, neither of which "checked out." Often this theorem suggests many candidates, as we see in Example 5. In such a case, we use technology and a variety of algebraic methods to locate the rational zeros.

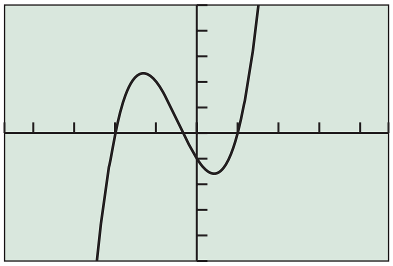
EXAMPLE 5 Finding the Rational Zeros

Find the rational zeros of $f(x) = 3x^3 + 4x^2 - 5x - 2$.

SOLUTION Because the leading coefficient is 3 and constant coefficient is -2, the Rational Zeros Theorem yields several potential rational zeros of f. We take an organized approach to our solution.

Potential Rational Zeros:

$$\frac{\text{Factors of } -2}{\text{Factors of } 3} : \frac{\pm 1, \pm 2}{\pm 1, \pm 3} : \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$$



[–4.7, 4.7] by [–10, 10]

FIGURE 2.37 The function $f(x) = 3x^3 + 4x^2 - 5x - 2$ has three real zeros. (Example 5)

Figure 2.37 suggests that, among our candidates, 1, -2, and possibly $-1/3$ or $-2/3$ are the most likely to be rational zeros. We use synthetic division because it tells us whether a number is a zero and, if so, how to factor the polynomial. To see whether 1 is a zero of f, we synthetically divide $f(x)$ by $x - 1$:

Zero of Divisor	1	3	4	−5	−2	Dividend
			3	7	2	
		3	7	2	0	Remainder
		Quotient				

Because the remainder is 0, $x - 1$ is a factor of $f(x)$ and 1 is a zero of f. By the division algorithm and factoring, we conclude

$$\begin{aligned} f(x) &= 3x^3 + 4x^2 - 5x - 2 \\ &= (x - 1)(3x^2 + 7x + 2) \\ &= (x - 1)(3x + 1)(x + 2) \end{aligned}$$

Therefore, the rational zeros of f are 1, $-1/3$, and -2. *Now try Exercise 35.*

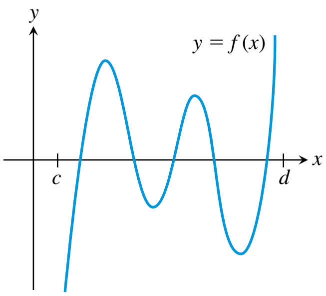


FIGURE 2.38 c is a lower bound and d is an upper bound for the real zeros of f.

Upper and Lower Bounds

We narrow our search for real zeros by using a test that identifies upper and lower bounds for real zeros. A number k is an **upper bound for the real zeros** of f if $f(x)$ is never zero when x is greater than k. On the other hand, a number k is a **lower bound for the real zeros** of f if $f(x)$ is never zero when x is less than k. So if c is a lower bound and d is an upper bound for the real zeros of a function f, all of the real zeros of f must lie in the interval $[c, d]$. Figure 2.38 illustrates this situation.

Upper and Lower Bound Tests for Real Zeros

Let f be a polynomial function of degree $n \geq 1$ with a positive leading coefficient. Suppose $f(x)$ is divided by $x - k$ using synthetic division.

- If $k \geq 0$ and every number in the last line is nonnegative (positive or zero), then k is an *upper bound* for the real zeros of f.
- If $k \leq 0$ and the numbers in the last line are alternately nonnegative and nonpositive, then k is a *lower bound* for the real zeros of f.

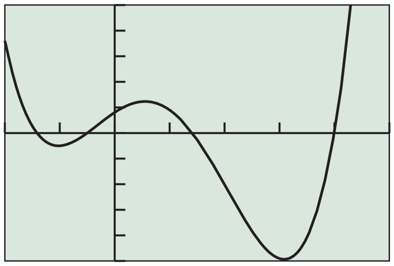
EXAMPLE 6 Establishing Bounds for Real Zeros

Prove that all of the real zeros of $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$ must lie in the interval $[-2, 5]$.

SOLUTION We must prove that 5 is an upper bound and -2 is a lower bound on the real zeros of f. The function f has a positive leading coefficient, so we employ the upper and lower bound tests, and use synthetic division:

5	2	−7	−8	14	8	
		10	15	35	245	
	2	3	7	49	253	Last line
−2	2	−7	−8	14	8	
		−4	22	−28	28	
	2	−11	14	−14	36	Last line

Because the last line in the first division scheme consists of all positive numbers, 5 is an upper bound. Because the last line in the second division consists of numbers of alternating signs, -2 is a lower bound. All of the real zeros of f must therefore lie in the closed interval $[-2, 5]$. ***Now try Exercise 37.***



[–2, 5] by [–50, 50]

FIGURE 2.39 $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$ has all of its real zeros in $[-2, 5]$. (Example 7)

EXAMPLE 7 Finding the Real Zeros of a Polynomial Function

Find all of the real zeros of $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$.

SOLUTION From Example 6 we know that all of the real zeros of f must lie in the closed interval $[-2, 5]$. So in Figure 2.39 we set our Xmin and Xmax accordingly.

Next we use the Rational Zeros Theorem.

Potential Rational Zeros:

$$\frac{\text{Factors of 8}}{\text{Factors of 2}} : \frac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1, \pm 2} : \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}$$

We compare the x-intercepts of the graph in Figure 2.39 and our list of candidates, and decide 4 and $-1/2$ are the only potential rational zeros worth pursuing.

4	2	−7	−8	14	8
		8	4	−16	−8
	2	1	−4	−2	0

From this first synthetic division we conclude

$$\begin{aligned} f(x) &= 2x^4 - 7x^3 - 8x^2 + 14x + 8 \\ &= (x - 4)(2x^3 + x^2 - 4x - 2) \end{aligned}$$

and we now divide the cubic factor $2x^3 + x^2 - 4x - 2$ by $x + 1/2$:

−1/2	2	1	−4	−2
		−1	0	2
	2	0	−4	0

(continued)

This second synthetic division allows us to complete the factoring of $f(x)$.

$$\begin{aligned} f(x) &= (x-4)(2x^3 + x^2 - 4x - 2) \\ &= (x-4)\left(x + \frac{1}{2}\right)(2x^2 - 4) \\ &= 2(x-4)\left(x + \frac{1}{2}\right)(x^2 - 2) \\ &= (x-4)(2x+1)(x+\sqrt{2})(x-\sqrt{2}) \end{aligned}$$

The zeros of f are the rational numbers 4 and $-1/2$ and the irrational numbers $-\sqrt{2}$ and $\sqrt{2}$. ***Now try Exercise 49.***

A polynomial function cannot have more real zeros than its degree, but it can have fewer. When a polynomial has fewer real zeros than its degree, the upper and lower bound tests help us know that we have found them all, as illustrated by Example 8.

EXAMPLE 8 Finding the Real Zeros of a Polynomial Function

Prove that all of the real zeros of $f(x) = 10x^5 - 3x^2 + x - 6$ lie in the interval $[0, 1]$, and find them.

SOLUTION We first prove that 1 is an upper bound and 0 is a lower bound for the real zeros of f. The function f has a positive leading coefficient, so we use synthetic division and the upper and lower bound tests:

$$\begin{array}{r|rrrrrrl} 1 & 10 & 0 & 0 & -3 & 1 & -6 & \\ & & 10 & 10 & 10 & 7 & 8 & \\ \hline & 10 & 10 & 10 & 7 & 8 & 2 & \text{Last line} \end{array}$$

$$\begin{array}{r|rrrrrrl} 0 & 10 & 0 & 0 & -3 & 1 & -6 & \\ & & 0 & 0 & 0 & 0 & 0 & \\ \hline & 10 & 0 & 0 & -3 & 1 & -6 & \text{Last line} \end{array}$$

Because the last line in the first division scheme consists of all nonnegative numbers, 1 is an upper bound. Because the last line in the second division consists of numbers that are alternately nonnegative and nonpositive, 0 is a lower bound. All of the real zeros of f must therefore lie in the closed interval $[0, 1]$. So in Figure 2.40 we set our Xmin and Xmax accordingly.

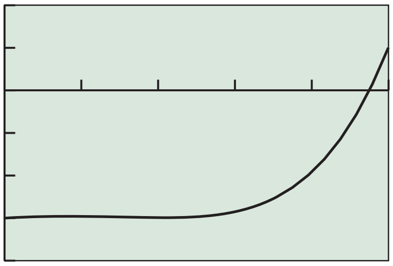
Next we use the Rational Zeros Theorem.

Potential Rational Zeros:

$$\frac{\text{Factors of } -6}{\text{Factors of } 10} : \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2, \pm 5, \pm 10} :$$

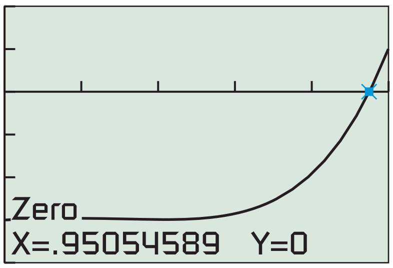
$$\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$$

We compare the x-intercepts of the graph in Figure 2.40 and our list of candidates, and decide f has no rational zeros. From Figure 2.40 we see that f changes sign on the interval $[0.8, 1]$. Thus by the Intermediate Value Theorem, f must have a real zero on this interval. Because it is not rational, we conclude that it is irrational. Figure 2.41 shows that this lone real zero of f is approximately 0.95. ***Now try Exercise 55.***



[0, 1] by [–8, 4]

FIGURE 2.40 $y = 10x^5 - 3x^2 + x - 6$. (Example 8)



[0, 1] by [–8, 4]

FIGURE 2.41 An approximation for the irrational zero of $f(x) = 10x^5 - 3x^2 + x - 6$. (Example 8)

QUICK REVIEW 2.4 *(For help, go to Sections A.2. and A.3.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, rewrite the expression as a polynomial in standard form.

1. $\dfrac{x^3 - 4x^2 + 7x}{x}$

2. $\dfrac{2x^3 - 5x^2 - 6x}{2x}$

3. $\dfrac{x^4 - 3x^2 + 7x^5}{x^2}$

4. $\dfrac{6x^4 - 2x^3 + 7x^2}{3x^2}$

In Exercises 5–10, factor the polynomial into linear factors.

5. $x^3 - 4x$

6. $6x^2 - 54$

7. $4x^2 + 8x - 60$

8. $15x^3 - 22x^2 + 8x$

9. $x^3 + 2x^2 - x - 2$

10. $x^4 + x^3 - 9x^2 - 9x$

SECTION 2.4 EXERCISES

In Exercises 1–6, divide $f(x)$ by $d(x)$, and write a summary statement in polynomial form and fraction form.

1. $f(x) = x^2 - 2x + 3;\ d(x) = x - 1$

2. $f(x) = x^3 - 1;\ d(x) = x + 1$

3. $f(x) = x^3 + 4x^2 + 7x - 9;\ d(x) = x + 3$

4. $f(x) = 4x^3 - 8x^2 + 2x - 1;\ d(x) = 2x + 1$

5. $f(x) = x^4 - 2x^3 + 3x^2 - 4x + 6;\ d(x) = x^2 + 2x - 1$

6. $f(x) = x^4 - 3x^3 + 6x^2 - 3x + 5;\ d(x) = x^2 + 1$

In Exercises 7–12, divide using synthetic division, and write a summary statement in fraction form.

7. $\dfrac{x^3 - 5x^2 + 3x - 2}{x + 1}$

8. $\dfrac{2x^4 - 5x^3 + 7x^2 - 3x + 1}{x - 3}$

9. $\dfrac{9x^3 + 7x^2 - 3x}{x - 10}$

10. $\dfrac{3x^4 + x^3 - 4x^2 + 9x - 3}{x + 5}$

11. $\dfrac{5x^4 - 3x + 1}{4 - x}$

12. $\dfrac{x^8 - 1}{x + 2}$

In Exercises 13–18, use the Remainder Theorem to find the remainder when $f(x)$ is divided by $x - k$.

13. $f(x) = 2x^2 - 3x + 1;\ k = 2$

14. $f(x) = x^4 - 5;\ k = 1$

15. $f(x) = x^3 - x^2 + 2x - 1;\ k = -3$

16. $f(x) = x^3 - 3x + 4;\ k = -2$

17. $f(x) = 2x^3 - 3x^2 + 4x - 7;\ k = 2$

18. $f(x) = x^5 - 2x^4 + 3x^2 - 20x + 3;\ k = -1$

In Exercises 19–24, use the Factor Theorem to determine whether the first polynomial is a factor of the second polynomial.

19. $x - 1;\ x^3 - x^2 + x - 1$

20. $x - 3;\ x^3 - x^2 - x - 15$

21. $x - 2;\ x^3 + 3x - 4$

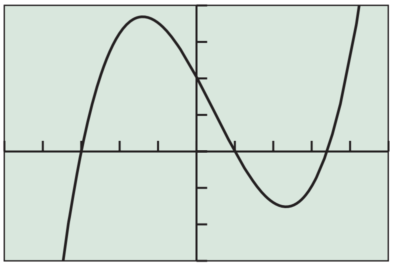
22. $x - 2;\ x^3 - 3x - 2$

23. $x + 2;\ 4x^3 + 9x^2 - 3x - 10$

24. $x + 1;\ 2x^{10} - x^9 + x^8 + x^7 + 2x^6 - 3$

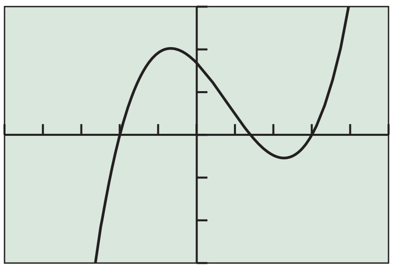
In Exercises 25 and 26, use the graph to guess possible linear factors of $f(x)$. Then completely factor $f(x)$ with the aid of synthetic division.

25. $f(x) = 5x^3 - 7x^2 - 49x + 51$



[–5, 5] by [–75, 100]

26. $f(x) = 5x^3 - 12x^2 - 23x + 42$



[–5, 5] by [–75, 75]

In Exercises 27–30, find the polynomial function with leading coefficient 2 that has the given degree and zeros.

27. Degree 3, with -2, 1, and 4 as zeros

28. Degree 3, with -1, 3, and -5 as zeros

29. Degree 3, with 2, $\frac{1}{2}$, and $\frac{3}{2}$ as zeros

30. Degree 4, with -3, -1, 0, and $\frac{5}{2}$ as zeros

In Exercises 31 and 32, using only algebraic methods, find the cubic function with the given table of values. Check with a grapher.

31.

x	-4	0	3	5
$f(x)$	0	180	0	0

32.

x	-2	-1	1	5
$f(x)$	0	24	0	0

In Exercises 33–36, use the Rational Zeros Theorem to write a list of all potential rational zeros. Then determine which ones, if any, are zeros.

33. $f(x) = 6x^3 - 5x - 1$

34. $f(x) = 3x^3 - 7x^2 + 6x - 14$

35. $f(x) = 2x^3 - x^2 - 9x + 9$

36. $f(x) = 6x^4 - x^3 - 6x^2 - x - 12$

In Exercises 37–40, use synthetic division to prove that the number k is an upper bound for the real zeros of the function f.

37. $k = 3;\ f(x) = 2x^3 - 4x^2 + x - 2$

38. $k = 5;\ f(x) = 2x^3 - 5x^2 - 5x - 1$

39. $k = 2;\ f(x) = x^4 - x^3 + x^2 + x - 12$

40. $k = 3;\ f(x) = 4x^4 - 6x^3 - 7x^2 + 9x + 2$

In Exercises 41–44, use synthetic division to prove that the number k is a lower bound for the real zeros of the function f.

41. $k = -1;\ f(x) = 3x^3 - 4x^2 + x + 3$

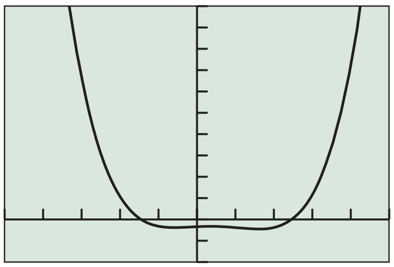
42. $k = -3;\ f(x) = x^3 + 2x^2 + 2x + 5$

43. $k = 0;\ f(x) = x^3 - 4x^2 + 7x - 2$

44. $k = -4;\ f(x) = 3x^3 - x^2 - 5x - 3$

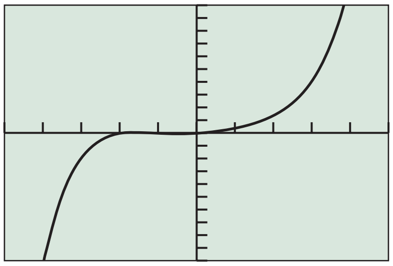
In Exercises 45–48, use the upper and lower bound tests to decide whether there could be real zeros for the function outside the window shown. If so, check for additional zeros.

45. $f(x) = 6x^4 - 11x^3 - 7x^2 + 8x - 34$



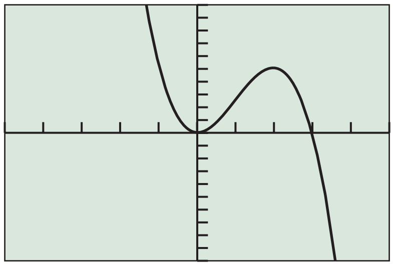
[–5, 5] by [–200, 1000]

46. $f(x) = x^5 - x^4 + 21x^2 + 19x - 3$



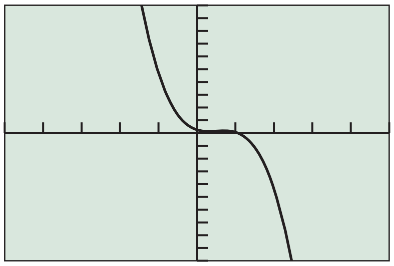
[–5, 5] by [–1000, 1000]

47. $f(x) = x^5 - 4x^4 - 129x^3 + 396x^2 - 8x + 3$



[–5, 5] by [–1000, 1000]

48. $f(x) = 2x^5 - 5x^4 - 141x^3 + 216x^2 - 91x + 25$



[–5, 5] by [–1000, 1000]

In Exercises 49–56, find all of the real zeros of the function, finding exact values whenever possible. Identify each zero as rational or irrational.

49. $f(x) = 2x^3 - 3x^2 - 4x + 6$

50. $f(x) = x^3 + 3x^2 - 3x - 9$

51. $f(x) = x^3 + x^2 - 8x - 6$

52. $f(x) = x^3 - 6x^2 + 7x + 4$

53. $f(x) = x^4 - 3x^3 - 6x^2 + 6x + 8$

54. $f(x) = x^4 - x^3 - 7x^2 + 5x + 10$

55. $f(x) = 2x^4 - 7x^3 - 2x^2 - 7x - 4$

56. $f(x) = 3x^4 - 2x^3 + 3x^2 + x - 2$

57. Setting Production Schedules The Sunspot Small Appliance Co. determines that the supply function for their EverCurl hair dryer is $S(p) = 6 + 0.001p^3$ and that its demand function is $D(p) = 80 - 0.02p^2$, where p is the price. Determine the price for which the supply equals the demand

and the number of hair dryers corresponding to this equilibrium price.

58. Setting Production Schedules The Pentkon Camera Co. determines that the supply and demand functions for their 35 mm–70 mm zoom lens are $S(p) = 200 - p + 0.000007p^4$ and $D(p) = 1500 - 0.0004p^3$, where p is the price. Determine the price for which the supply equals the demand and the number of zoom lenses corresponding to this equilibrium price.

59. Find the remainder when $x^{40} - 3$ is divided by $x + 1$.

60. Find the remainder when $x^{63} - 17$ is divided by $x - 1$.

61. Let $f(x) = x^4 + 2x^3 - 11x^2 - 13x + 38$.

(a) Use the upper and lower bound tests to prove that all of the real zeros of f lie on the interval $[-5, 4]$.

(b) Find all of the rational zeros of f.

(c) Factor $f(x)$ using the rational zero(s) found in (b).

(d) Approximate all of the irrational zeros of f.

(e) Use synthetic division and the irrational zero(s) found in (d) to continue the factorization of $f(x)$ begun in (c).

62. Lewis's distance D from a motion detector is given by the data in Table 2.15.

Table 2.15 Motion Detector Data

t (sec)	D (m)	t (sec)	D (m)
0.0	1.00	4.5	0.99
0.5	1.46	5.0	0.84
1.0	1.99	5.5	1.28
1.5	2.57	6.0	1.87
2.0	3.02	6.5	2.58
2.5	3.34	7.0	3.23
3.0	2.91	7.5	3.78
3.5	2.31	8.0	4.40
4.0	1.57		

(a) Find a cubic regression model, and graph it together with a scatter plot of the data.

(b) Use the cubic regression model to estimate how far Lewis is from the motion detector initially.

(c) Use the cubic regression model to estimate when Lewis changes direction. How far from the motion detector is he when he changes direction?

Standardized Test Questions

63. True or False The polynomial function $f(x)$ has a factor $x + 2$ if and only if $f(2) = 0$. Justify your answer.

64. True or False If $f(x) = (x - 1)(2x^2 - x + 1) + 3$, then the remainder when $f(x)$ is divided by $x - 1$ is 3. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve the problem.

65. Multiple Choice Let f be a polynomial function with $f(3) = 0$. Which of the following statements is not true?

(A) $x + 3$ is a factor of $f(x)$. **(B)** $x - 3$ is a factor of $f(x)$.

(C) $x = 3$ is a zero of $f(x)$. **(D)** 3 is an x-intercept of $f(x)$.

(E) The remainder when $f(x)$ is divided by $x - 3$ is zero.

66. Multiple Choice Let $f(x) = 2x^3 + 7x^2 + 2x - 3$. Which of the following is not a possible rational root of f?

(A) -3 **(B)** -1 **(C)** 1 **(D)** 1/2 **(E)** 2/3

67. Multiple Choice Let $f(x) = (x + 2)(x^2 + x - 1) - 3$. Which of the following statements is not true?

(A) The remainder when $f(x)$ is divided by $x + 2$ is -3.

(B) The remainder when $f(x)$ is divided by $x - 2$ is -3.

(C) The remainder when $f(x)$ is divided by $x^2 + x - 1$ is -3.

(D) $x + 2$ is not a factor of $f(x)$.

(E) $f(x)$ is not evenly divisible by $x + 2$.

68. Multiple Choice Let $f(x) = (x^2 + 1)(x - 2) + 7$. Which of the following statements is not true?

(A) The remainder when $f(x)$ is divided by $x^2 + 1$ is 7.

(B) The remainder when $f(x)$ is divided by $x - 2$ is 7.

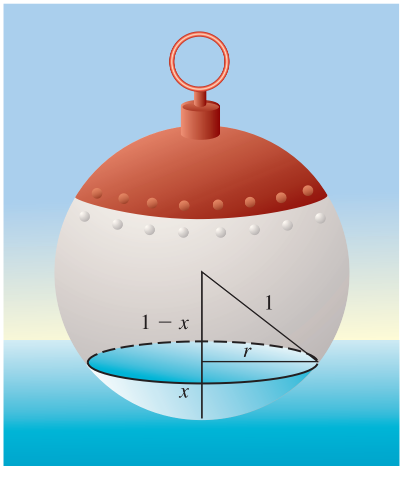
(C) $f(2) = 7$ **(D)** $f(0) = 5$

(E) f does not have a real root.

Explorations

69. Archimedes' Principle A spherical buoy has a radius of 1 m and a density one-fourth that of seawater. By Archimedes' Principle, the weight of the displaced water will equal the weight of the buoy.

- Let x = the depth to which the buoy sinks.
- Let d = the density of seawater.
- Let r = the radius of the circle formed where buoy, air, and water meet. See the figure below.



Notice in the figure that $r^2 = 1 - (1 - x)^2 = 2x - x^2$, and recall from geometry that the volume of submerged *spherical cap* is $V = \frac{\pi x}{6} \cdot (3r^2 + x^2)$.

(a) Verify that the volume of the buoy is $4\pi/3$.

(b) Use your result from (a) to establish the weight of the buoy as $\pi d/3$.

(c) Prove the weight of the displaced water is $\pi d \cdot x(3r^2 + x^2)/6$.

(d) Approximate the depth to which the buoy will sink.

70. Archimedes' Principle Using the scenario of Exercise 69, find the depth to which the buoy will sink if its density is one-fifth that of seawater.

71. Biological Research Stephanie, a biologist who does research for the poultry industry, models the population P of wild turkeys, t days after being left to reproduce, with the function

$$P(t) = -0.00001t^3 + 0.002t^2 + 1.5t + 100.$$

(a) Graph the function $y = P(t)$ for appropriate values of t.

(b) Find what the maximum turkey population is and when it occurs.

(c) Assuming that this model continues to be accurate, when will this turkey population become extinct?

(d) Writing to Learn Create a scenario that could explain the growth exhibited by this turkey population.

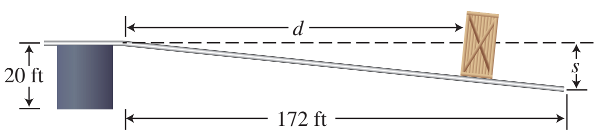
72. Architectural Engineering Dave, an engineer at the Trumbauer Group, Inc., an architectural firm, completes structural specifications for a 172-ft-long steel beam, anchored at one end to a piling 20 ft above the ground. He knows that when a 200-lb object is placed d feet from the anchored end, the beam bends s feet where

$$s = (3 \times 10^{-7})d^2(550 - d).$$

(a) What is the independent variable in this polynomial function?

(b) What are the dimensions of a viewing window that shows a graph for the values that make sense in this problem situation?

(c) How far is the 200-lb object from the anchored end if the vertical deflection is 1.25 ft?



73. A classic theorem, **Descartes' Rule of Signs**, tells us about the number of positive and negative real zeros of a polynomial function, by looking at the polynomial's variations in sign. A *variation in sign* occurs when consecutive coefficients (in standard form) have opposite signs.

If $f(x) = a_n x^n + \cdots + a_0$ is a polynomial of degree n, then

- The number of positive real zeros of f is equal to the number of variations in sign of $f(x)$, or that number less some even number.
- The number of negative real zeros of f is equal to the number of variations in sign of $f(-x)$, or that number less some even number.

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.

(a) $f(x) = x^3 + x^2 - x + 1$

(b) $f(x) = x^3 + x^2 + x + 1$

(c) $f(x) = 2x^3 + x - 3$

(d) $g(x) = 5x^4 + x^2 - 3x - 2$

Extending the Ideas

74. Writing to Learn Graph each side of the Example 3 summary equation:

$$f(x) = \frac{2x^3 - 3x^2 - 5x - 12}{x - 3} \text{ and}$$

$$g(x) = 2x^2 + 3x + 4, \quad x \neq 3$$

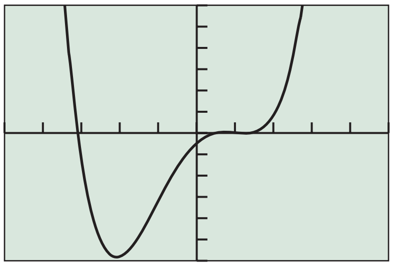
How are these functions related? Include a discussion of the domain and continuity of each function.

75. Writing to Learn Explain how to carry out the following division using synthetic division. Work through the steps with complete explanations. Interpret and check your result.

$$\frac{4x^3 - 5x^2 + 3x + 1}{2x - 1}$$

76. Writing to Learn The figure shows a graph of $f(x) = x^4 + 0.1x^3 - 6.5x^2 + 7.9x - 2.4$. Explain how to use a grapher to justify the statement.

$$f(x) = x^4 + 0.1x^3 - 6.5x^2 + 7.9x - 2.4$$
$$\approx (x + 3.10)(x - 0.5)(x - 1.13)(x - 1.37)$$



[–5, 5] by [–30, 30]

77. **(a)** **Writing to Learn** Write a paragraph that describes how the zeros of $f(x) = (1/3)x^3 + x^2 + 2x - 3$ are related to the zeros of $g(x) = x^3 + 3x^2 + 6x - 9$. In what ways does this example illustrate how the Rational Zeros Theorem can be applied to find the zeros of a polynomial with *rational* number coefficients?

(b) Find the rational zeros of $f(x) = x^3 - \frac{7}{6}x^2 - \frac{20}{3}x + \frac{7}{2}$.

(c) Find the rational zeros of $f(x) = x^3 - \frac{5}{2}x^2 - \frac{37}{12}x + \frac{5}{2}$.

78. Use the Rational Zeros Theorem to prove $\sqrt{2}$ is irrational.

79. **Group Activity** *Work in groups of three.* Graph $f(x) = x^4 + x^3 - 8x^2 - 2x + 7$.

(a) Use grapher methods to find approximate real number zeros.

(b) Identify a list of four linear factors whose product could be called an *approximate factorization of* $f(x)$.

(c) Discuss what graphical and numerical methods you could use to show that the factorization from part (b) is reasonable.