2.3 Polynomial Functions of Higher Degree with Modeling

What you'll learn about

- Graphs of Polynomial Functions
- End Behavior of Polynomial Functions
- Zeros of Polynomial Functions
- Intermediate Value Theorem
- Modeling

... and why

These topics are important in modeling and can be used to provide approximations to more complicated functions, as you will see if you study calculus.

Graphs of Polynomial Functions

As we saw in Section 2.1, a polynomial function of degree 0 is a constant function and graphs as a horizontal line. A polynomial function of degree 1 is a linear function; its graph is a slant line. A polynomial function of degree 2 is a quadratic function; its graph is a parabola.

We now consider polynomial functions of higher degree. These include **cubic functions** (polynomials of degree 3) and **quartic functions** (polynomials of degree 4). Recall that a polynomial function of degree n can be written in the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,\ a_n \neq 0.$$

Here are some important definitions associated with polynomial functions and this equation.

DEFINITION The Vocabulary of Polynomials

- Each monomial in this sum—$a_n x^n, a_{n-1}x^{n-1}, \ldots, a_0$—is a **term** of the polynomial.
- A polynomial function written in this way, with terms in descending degree, is written in **standard form**.
- The constants $a_n, a_{n-1}, \ldots, a_0$ are the **coefficients** of the polynomial.
- The term $a_n x^n$ is the **leading term**, and a_0 is the constant term.

In Example 1 we use the fact from Section 2.1 that the constant term a_0 of a polynomial function p is both the initial value of the function $p(0)$ and the y-intercept of the graph to provide a quick and easy check of the transformed graphs.

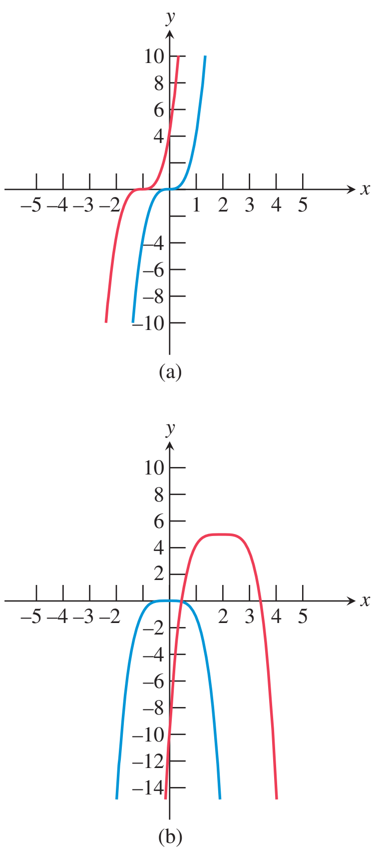


FIGURE 2.19 (a) The graphs of $g(x) = 4(x + 1)^3$ and $f(x) = 4x^3$. (b) The graphs of $h(x) = -(x - 2)^4 + 5$ and $f(x) = -x^4$. (Example 1)

EXAMPLE 1 Graphing Transformations of Monomial Functions

Describe how to transform the graph of an appropriate monomial function $f(x) = a_n x^n$ into the graph of the given function. Sketch the transformed graph by hand and support your answer with a grapher. Compute the location of the y-intercept as a check on the transformed graph.

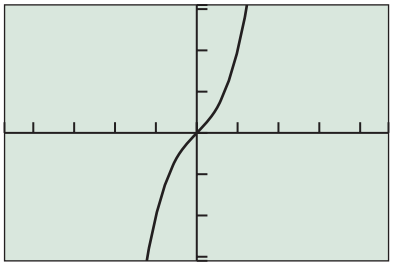
(a) $g(x) = 4(x + 1)^3$ **(b)** $h(x) = -(x - 2)^4 + 5$

SOLUTION

(a) You can obtain the graph of $g(x) = 4(x + 1)^3$ by shifting the graph of $f(x) = 4x^3$ one unit to the left, as shown in Figure 2.19a. The y-intercept of the graph of g is $g(0) = 4(0 + 1)^3 = 4$, which appears to agree with the transformed graph.

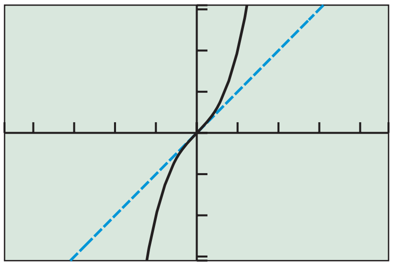
(b) You can obtain the graph of $h(x) = -(x - 2)^4 + 5$ by shifting the graph of $f(x) = -x^4$ two units to the right and five units up, as shown in Figure 2.19b. The y-intercept of the graph of h is $h(0) = -(0 - 2)^4 + 5 = -16 + 5 = -11$, which appears to agree with the transformed graph.

Now try Exercise 1.



[–4.7, 4.7] by [–3.1, 3.1]

(a)



[–4.7, 4.7] by [–3.1, 3.1]

(b)

FIGURE 2.20 The graph of $f(x) = x^3 + x$ (a) by itself and (b) with $y = x$. (Example 2a)

Example 2 shows what can happen when simple monomial functions are combined to obtain polynomial functions. The resulting polynomials are *not* mere translations of monomials.

EXAMPLE 2 Graphing Combinations of Monomial Functions

Graph the polynomial function, locate its extrema and zeros, and explain how it is related to the monomials from which it is built.

(a) $f(x) = x^3 + x$ **(b)** $g(x) = x^3 - x$

SOLUTION

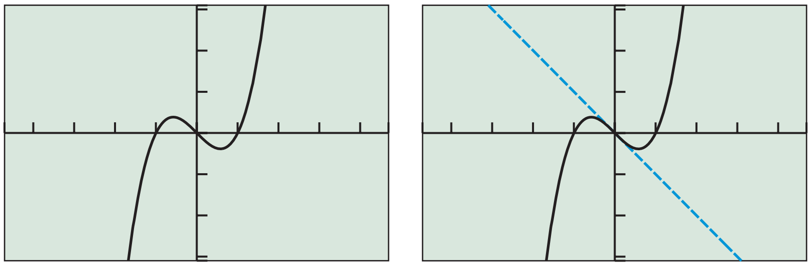
(a) The graph of $f(x) = x^3 + x$ is shown in Figure 2.20a. The function f is increasing on $(-\infty, \infty)$, with no extrema. The function factors as $f(x) = x(x^2 + 1)$ and has one zero at $x = 0$.

The general shape of the graph is much like the graph of its leading term x^3, but near the origin f behaves much like its other term x, as shown in Figure 2.20b. The function f is odd, just like its two building block monomials.

(b) The graph of $g(x) = x^3 - x$ is shown in Figure 2.21a. The function g has a local maximum of about 0.38 at $x \approx -0.58$ and a local minimum of about -0.38 at $x \approx 0.58$. The function factors as $g(x) = x(x + 1)(x - 1)$ and has zeros located at $x = -1$, $x = 0$, and $x = 1$.

The general shape of the graph is much like the graph of its leading term x^3, but near the origin g behaves much like its other term $-x$, as shown in Figure 2.21b. The function g is odd, just like its two building block monomials.

Now try Exercise 7.



[–4.7, 4.7] by [–3.1, 3.1]

(a)

[–4.7, 4.7] by [–3.1, 3.1]

(b)

FIGURE 2.21 The graph of $g(x) = x^3 - x$ (a) by itself and (b) with $y = -x$. (Example 2b)

We have now seen a few examples of graphs of polynomial functions, but are these typical? What do graphs of polynomials look like in general?

To begin our answer, let's first recall that every polynomial function is defined and continuous for all real numbers. Not only are graphs of polynomials unbroken without jumps or holes, but they are *smooth*, unbroken lines or curves, with no sharp corners or cusps. Typical graphs of cubic and quartic functions are shown in Figures 2.22 and 2.23.

Imagine horizontal lines passing through the graphs in Figures 2.22 and 2.23, acting as x-axes. Each intersection would be an x-intercept that would correspond to a zero of the function. From this mental experiment, we see that cubic functions have at most three zeros and quartic functions have at most four zeros. Focusing on the high and low points in Figures 2.22 and 2.23, we see that cubic functions have at most two local extrema and quartic functions have at most three local extrema. These observations generalize in the following way:

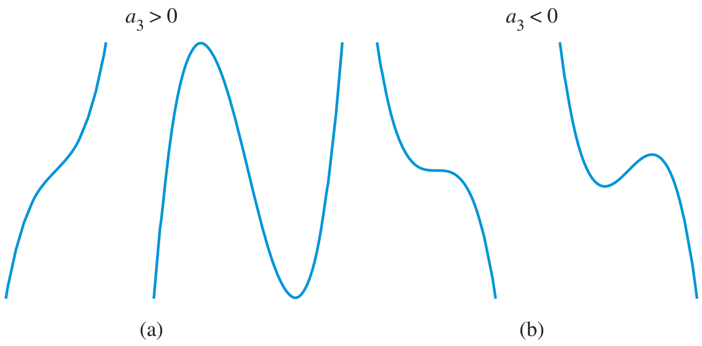


FIGURE 2.22 Graphs of four typical cubic functions: (a) two with positive and (b) two with negative leading coefficients.

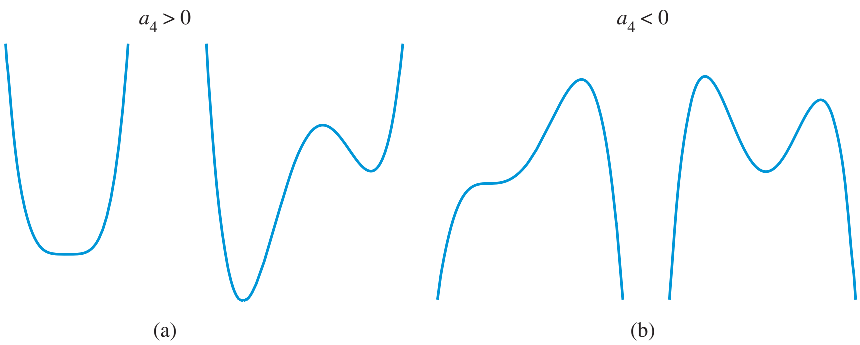


FIGURE 2.23 Graphs of four typical quartic functions: (a) two with positive and (b) two with negative leading coefficients.

THEOREM Local Extrema and Zeros of Polynomial Functions

A polynomial function of degree n has at most $n - 1$ local extrema and at most n zeros.

End Behavior of Polynomial Functions

An important characteristic of polynomial functions is their end behavior. As we shall see, the end behavior of a polynomial is closely related to the end behavior of its leading term. Exploration 1 examines the end behavior of monomial functions, which are potential leading terms for polynomial functions.

Technology Note

For a cubic, when you change the horizontal window by a factor of k, it usually is a good idea to change the vertical window by a factor of k^3. Similar statements can be made about polynomials of other degrees.

EXPLORATION 1 Investigating the End Behavior of $f(x) = a_n x^n$

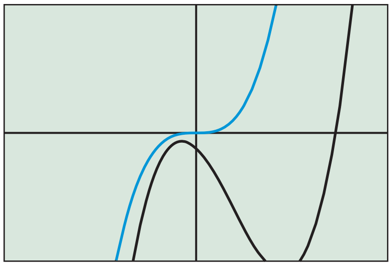
Graph each function in the window $[-5, 5]$ by $[-15, 15]$. Describe the end behavior using $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

1. **(a)** $f(x) = 2x^3$ **(b)** $f(x) = -x^3$
(c) $f(x) = x^5$ **(d)** $f(x) = -0.5x^7$

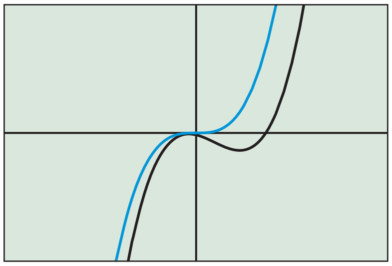
2. **(a)** $f(x) = -3x^4$ **(b)** $f(x) = 0.6x^4$
(c) $f(x) = 2x^6$ **(d)** $f(x) = -0.5x^2$

3. **(a)** $f(x) = -0.3x^5$ **(b)** $f(x) = -2x^2$
(c) $f(x) = 3x^4$ **(d)** $f(x) = 2.5x^3$

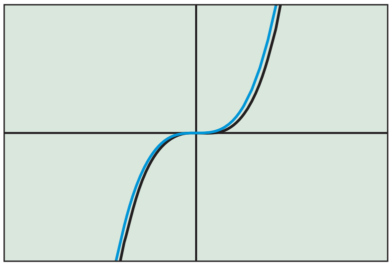
Describe the patterns you observe. In particular, how do the values of the coefficient a_n and the degree n affect the end behavior of $f(x) = a_n x^n$?



[–7, 7] by [–25, 25]
(a)



[–14, 14] by [–200, 200]
(b)



[–56, 56] by [–12800, 12800]
(c)

FIGURE 2.24 As the viewing window gets larger, the graphs of $f(x) = x^3 - 4x^2 - 5x - 3$ and $g(x) = x^3$ look more and more alike. (Example 3)

Example 3 illustrates the link between the end behavior of a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ and its leading term $a_n x^n$.

EXAMPLE 3 Comparing the Graphs of a Polynomial and Its Leading Term

Superimpose the graphs of $f(x) = x^3 - 4x^2 - 5x - 3$ and $g(x) = x^3$ in successively larger viewing windows, a process called **zoom out**. Continue zooming out until the graphs look nearly identical.

SOLUTION

Figure 2.24 shows three views of the graphs of $f(x) = x^3 - 4x^2 - 5x - 3$ and $g(x) = x^3$ in progressively larger viewing windows. As the dimensions of the window increase, it gets harder to tell them apart. Moreover,

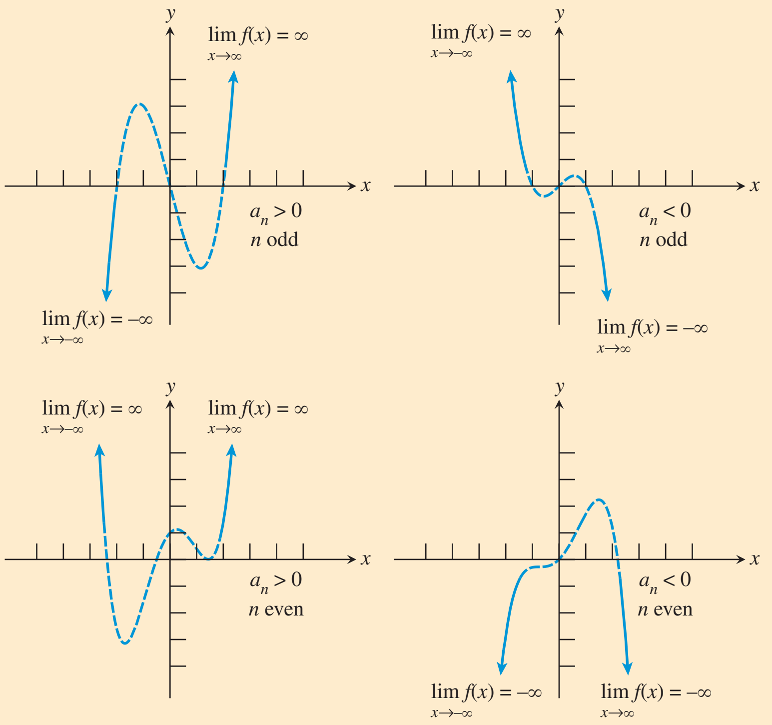
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x) = -\infty.$$

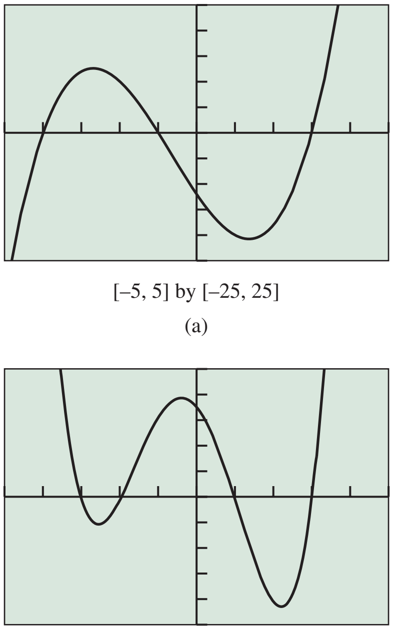
Now try Exercise 13.

Example 3 illustrates something that is true for all polynomials: *In sufficiently large viewing windows, the graph of a polynomial and the graph of its leading term appear to be identical.* Said another way, the leading term *dominates* the behavior of the polynomial as $|x| \to \infty$. Based on this fact and what we have seen in Exploration 1, there are four possible end behavior patterns for a polynomial function. The power and coefficient of the leading term tell us which one of the four patterns occurs.

Leading Term Test for Polynomial End Behavior

For any polynomial function $f(x) = a_n x^n + \cdots + a_1 x + a_0$, the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are determined by the degree n of the polynomial and its leading coefficient a_n:





[–5, 5] by [–25, 25]

(a)

[–5, 5] by [–50, 50]

(b)

FIGURE 2.25 (a) $f(x) = x^3 + 2x^2 - 11x - 12$, (b) $g(x) = 2x^4 + 2x^3 - 22x^2 - 18x + 35$. (Example 4)

EXAMPLE 4 Applying Polynomial Theory

Graph the polynomial in a window showing its extrema and zeros and its end behavior. Describe the end behavior using limits.

(a) $f(x) = x^3 + 2x^2 - 11x - 12$

(b) $g(x) = 2x^4 + 2x^3 - 22x^2 - 18x + 35$

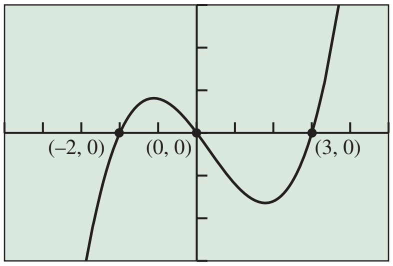
SOLUTION

(a) The graph of $f(x) = x^3 + 2x^2 - 11x - 12$ is shown in Figure 2.25a. The function f has 2 extrema and 3 zeros, the maximum number possible for a cubic. $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

(b) The graph of $g(x) = 2x^4 + 2x^3 - 22x^2 - 18x + 35$ is shown in Figure 2.25b. The function g has 3 extrema and 4 zeros, the maximum number possible for a quartic. $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = \infty$. ***Now try Exercise 19.***

Zeros of Polynomial Functions

Recall that finding the real number zeros of a function f is equivalent to finding the x-intercepts of the graph of $y = f(x)$ or the solutions to the equation $f(x) = 0$. Example 5 illustrates that *factoring* a polynomial function makes solving these three related problems an easy matter.



[–5, 5] by [–15, 15]

FIGURE 2.26 The graph of $y = x^3 - x^2 - 6x$, showing the three x-intercepts. (Example 5)

EXAMPLE 5 Finding the Zeros of a Polynomial Function

Find the zeros of $f(x) = x^3 - x^2 - 6x$.

SOLUTION

Solve Algebraically

We solve the related equation $f(x) = 0$ by factoring:

$$
\begin{aligned}
x^3 - x^2 - 6x &= 0 \\
x(x^2 - x - 6) &= 0 && \text{Remove common factor } x. \\
x(x - 3)(x + 2) &= 0 && \text{Factor quadratic.} \\
x = 0,\ x - 3 = 0, \quad \text{or} \quad x + 2 &= 0 && \text{Zero factor property} \\
x = 0,\ x = 3, \quad \text{or} \quad x &= -2
\end{aligned}
$$

So the zeros of f are 0, 3, and -2.

Support Graphically

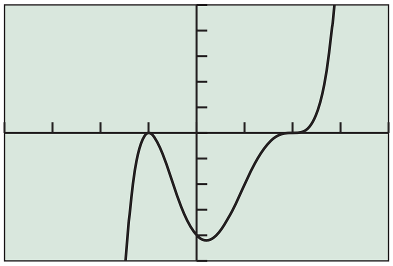
Use the features of your calculator to approximate the zeros of f. Figure 2.26 shows that there are three values. Based on our algebraic solution we can be sure that these values are exact. ***Now try Exercise 33.***

From Example 5, we see that if a polynomial function f is presented in factored form, each factor $(x - k)$ corresponds to a zero $x = k$, and if k is a real number, $(k, 0)$ is an x-intercept of the graph of $y = f(x)$.

When a factor is repeated, as in $f(x) = (x - 2)^3(x + 1)^2$, we say the polynomial function has a *repeated zero*. The function f has two repeated zeros. Because the factor $x - 2$ occurs three times, 2 is a zero of *multiplicity* 3. Similarly, -1 is a zero of multiplicity 2. The following definition generalizes this concept.

DEFINITION Multiplicity of a Zero of a Polynomial Function

If f is a polynomial function and $(x - c)^m$ is a factor of f but $(x - c)^{m+1}$ is not, then c is a zero of **multiplicity *m*** of f.



[–4, 4] by [–10, 10]

FIGURE 2.27 The graph of $f(x) = (x - 2)^3(x + 1)^2$, showing the x-intercepts.

A zero of multiplicity $m \geq 2$ is a **repeated zero**. Notice in Figure 2.27 that the graph of f just *kisses* the x-axis without crossing it at $(-1, 0)$, but that the graph of f crosses the x-axis at $(2, 0)$. This too can be generalized.

Zeros of Odd and Even Multiplicity

If a polynomial function f has a real zero c of odd multiplicity, then the graph of f crosses the x-axis at $(c, 0)$ and the value of f changes sign at $x = c$.
If a polynomial function f has a real zero c of even multiplicity, then the graph of f does not cross the x-axis at $(c, 0)$ and the value of f does not change sign at $x = c$.

In Example 5 none of the zeros were repeated. Because a nonrepeated zero has multiplicity 1, and 1 is odd, the graph of a polynomial function crosses the x-axis and has a sign change at every nonrepeated zero (Figure 2.26). Knowing where a graph crosses the x-axis and where it doesn't is important in curve sketching and in solving inequalities.

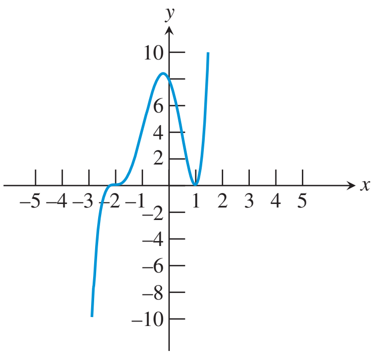


FIGURE 2.28 A sketch of the graph of $f(x) = (x + 2)^3(x - 1)^2$ showing the x-intercepts.

EXAMPLE 6 Sketching the Graph of a Factored Polynomial

State the degree and list the zeros of the function $f(x) = (x + 2)^3(x - 1)^2$. State the multiplicity of each zero and whether the graph crosses the x-axis at the corresponding x-intercept. Then sketch the graph of f by hand.

SOLUTION The degree of f is 5 and the zeros are $x = -2$ and $x = 1$. The graph crosses the x-axis at $x = -2$ because the multiplicity 3 is odd. The graph does not cross the x-axis at $x = 1$ because the multiplicity 2 is even. Notice that values of f are positive for $x > 1$, positive for $-2 < x < 1$, and negative for $x < -2$. Figure 2.28 shows a sketch of the graph of f. ***Now try Exercise 39.***

Intermediate Value Theorem

The *Intermediate Value Theorem* tells us that a sign change implies a real zero.

THEOREM Intermediate Value Theorem

If a and b are real numbers with $a < b$ and if f is continuous on the interval $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some number c in $[a, b]$.
In particular, if $f(a)$ and $f(b)$ have opposite signs (i.e., one is negative and the other is positive), then $f(c) = 0$ for some number c in $[a, b]$ (Figure 2.29).

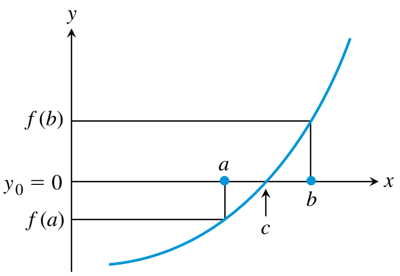


FIGURE 2.29 If $f(a) < 0 < f(b)$ and f is continuous on $[a, b]$, then there is a zero $x = c$ between a and b.

EXAMPLE 7 Using the Intermediate Value Theorem

Explain why a polynomial function of odd degree has at least one real zero.

SOLUTION Let f be a polynomial function of odd degree. Because f is odd, the leading term test tells us that $\lim_{x\to\infty} f(x) = -\lim_{x\to-\infty} f(x)$. So there exist real numbers a and b with $a < b$ and such that $f(a)$ and $f(b)$ have opposite signs. Because every polynomial function is defined and continuous for all real numbers, f is continuous on the interval $[a, b]$. Therefore, by the Intermediate Value Theorem, $f(c) = 0$ for some number c in $[a, b]$, and thus c is a real zero of f. ***Now try Exercise 61.***

In practice, the Intermediate Value Theorem is used in combination with our other mathematical knowledge and technological know-how.

Exact vs. Approximate

In Example 8, note that $x = 0.50$ is an exact answer; the others are approximate. Use by-hand substitution to confirm that $x = 1/2$ is an exact real zero.

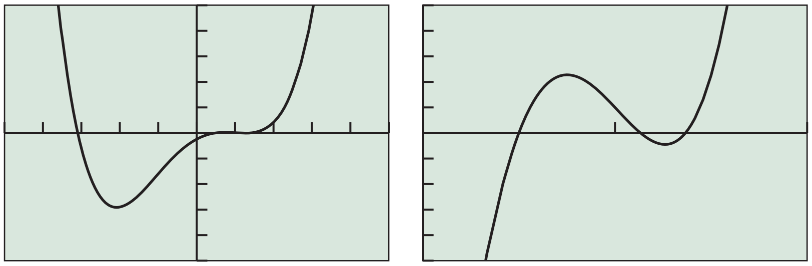
EXAMPLE 8 Zooming to Uncover Hidden Behavior

Find all of the real zeros of $f(x) = x^4 + 0.1x^3 - 6.5x^2 + 7.9x - 2.4$.

SOLUTION

Solve Graphically

Because f is of degree 4, there are at most four zeros. The graph in Figure 2.30a suggests a single zero (multiplicity 1) around $x = -3$ and a triple zero (multiplicity 3) around $x = 1$. Closer inspection around $x = 1$ in Figure 2.30b reveals three separate zeros. Using the grapher, we find the four zeros to be $x \approx 1.37$, $x \approx 1.13$, $x = 0.50$, and $x \approx -3.10$. (See the margin note.) *Now try Exercise 75.*



[–5, 5] by [–50, 50]

(a)

[0, 2] by [–0.5, 0.5]

(b)

FIGURE 2.30 Two views of $f(x) = x^4 + 0.1x^3 - 6.5x^2 + 7.9x - 2.4$. (Example 8)

Modeling

In the problem-solving process presented in Section 1.1, step 2 is to develop a mathematical model of the problem. When the model being developed is a polynomial function of higher degree, the algebraic and geometric thinking required can be rather involved. In solving Example 9 you may find it helpful to make a physical model out of paper or cardboard.

EXAMPLE 9 Designing a Box

Dixie Packaging Company has contracted to make boxes with a volume of approximately 484 in.^{3}. Squares are to be cut from the corners of a 20-in. by 25-in. piece of cardboard, and the flaps folded up to make an open box. (See Figure 2.31.) What size squares should be cut from the cardboard?

SOLUTION

Model

We know that the volume V = height $\times$ length $\times$ width. So let

$$x = \text{edge of cut-out square (height of box)}$$
$$25 - 2x = \text{length of the box}$$
$$20 - 2x = \text{width of the box}$$
$$V = x(25 - 2x)(20 - 2x)$$

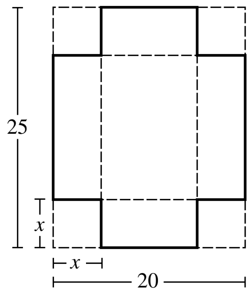


FIGURE 2.31

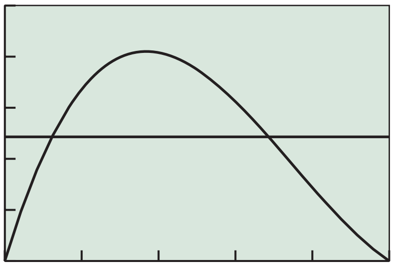
Solve Numerically and Graphically

For a volume of 484, we solve the equation $x(25 - 2x)(20 - 2x) = 484$. Because the width of the cardboard is 20 in., $0 \le x \le 10$. We use the table in Figure 2.32 to get a sense of the volume values to set the window for the graph in Figure 2.33. The cubic volume function intersects the constant volume of 484 at $x \approx 1.22$ and $x \approx 6.87$.

X	Y1	
1	414	
2	672	
3	798	
4	816	
5	750	
6	624	
7	462	

Y1 = X(20−2X)(25−...

FIGURE 2.32 A table to get a feel for the volume values in Example 9.



[0, 10] by [0, 1000]

FIGURE 2.33 $y_1 = x(25 - 2x)(20 - 2x)$ and $y_2 = 484$. (Example 9)

(continued)

Interpret

Squares with lengths of approximately 1.22 in. or 6.87 in. should be cut from the cardboard to produce a box with a volume of 484 in.^{3}. ***Now try Exercise 67.***

Just as any two points in the Cartesian plane with different x-values and different y-values determine a unique slant line and its related linear function, any three non-collinear points with different x-values determine a quadratic function. In general, $(n + 1)$ points positioned with sufficient generality determine a polynomial function of degree n. The process of fitting a polynomial of degree n to $(n + 1)$ points is **polynomial interpolation**. Exploration 2 involves two polynomial interpolation problems.

EXPLORATION 2 Interpolating Points with a Polynomial

1. Use cubic regression to fit a curve through the four points given in the table.

x	-2	1	3	8
y	2	0.5	-0.2	1.25

2. Use quartic regression to fit a curve through the five points given in the table.

x	3	4	5	6	8
y	-2	-4	-1	8	3

How good is the fit in each case? Why?

Generally we want a reason beyond "it fits well" to choose a model for genuine data. However, when no theoretical basis exists for picking a model, a balance between goodness of fit and simplicity of model is sought. For polynomials, we try to pick a model with the lowest possible degree that has a reasonably good fit.

QUICK REVIEW 2.3 *(For help, go to Sections A.2. and P.5.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–6, factor the polynomial into linear factors.

1. $x^2 - x - 12$
2. $x^2 - 11x + 28$
3. $3x^2 - 11x + 6$
4. $6x^2 - 5x + 1$
5. $3x^3 - 5x^2 + 2x$
6. $6x^3 - 22x^2 + 12x$

In Exercises 7–10, solve the equation *mentally*.

7. $x(x - 1) = 0$
8. $x(x + 2)(x - 5) = 0$
9. $(x + 6)^3(x + 3)(x - 1.5) = 0$
10. $(x + 6)^2(x + 4)^4(x - 5)^3 = 0$

SECTION 2.3 EXERCISES

In Exercises 1–6, describe how to transform the graph of an appropriate monomial function $f(x) = x^n$ into the graph of the given polynomial function. Sketch the transformed graph by hand and support your answer with a grapher. Compute the location of the y-intercept as a check on the transformed graph.

1. $g(x) = 2(x - 3)^3$ **2.** $g(x) = -(x + 5)^3$

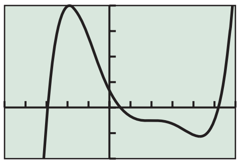
3. $g(x) = -\frac{1}{2}(x + 1)^3 + 2$ **4.** $g(x) = \frac{2}{3}(x - 3)^3 + 1$

5. $g(x) = -2(x + 2)^4 - 3$ **6.** $g(x) = 3(x - 1)^4 - 2$

In Exercises 7 and 8, graph the polynomial function, locate its extrema and zeros, and explain how it is related to the monomials from which it is built.

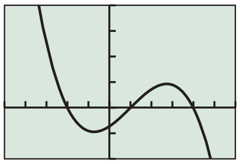
7. $f(x) = -x^4 + 2x$ **8.** $g(x) = 2x^4 - 5x^2$

In Exercises 9–12, match the polynomial function with its graph. Explain your choice. Do not use a graphing calculator.



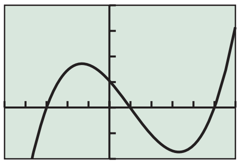
[–5, 6] by [–200, 400]

(a)



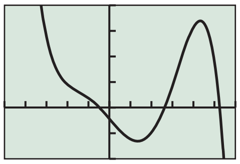
[–5, 6] by [–200, 400]

(b)



[–5, 6] by [–200, 400]

(c)



[–5, 6] by [–200, 400]

(d)

9. $f(x) = 7x^3 - 21x^2 - 91x + 104$

10. $f(x) = -9x^3 + 27x^2 + 54x - 73$

11. $f(x) = x^5 - 8x^4 + 9x^3 + 58x^2 - 164x + 69$

12. $f(x) = -x^5 + 3x^4 + 16x^3 - 2x^2 - 95x - 44$

In Exercises 13–16, graph the function pairs in the same series of viewing windows. Zoom out until the two graphs look nearly identical and state your final viewing window.

13. $f(x) = x^3 - 4x^2 - 5x - 3$ and $g(x) = x^3$

14. $f(x) = x^3 + 2x^2 - x + 5$ and $g(x) = x^3$

15. $f(x) = 2x^3 + 3x^2 - 6x - 15$ and $g(x) = 2x^3$

16. $f(x) = 3x^3 - 12x + 17$ and $g(x) = 3x^3$

In Exercises 17–24, graph the function in a viewing window that shows all of its extrema and x-intercepts. Describe the end behavior using limits.

17. $f(x) = (x - 1)(x + 2)(x + 3)$

18. $f(x) = (2x - 3)(4 - x)(x + 1)$

19. $f(x) = -x^3 + 4x^2 + 31x - 70$

20. $f(x) = x^3 - 2x^2 - 41x + 42$

21. $f(x) = (x - 2)^2(x + 1)(x - 3)$

22. $f(x) = (2x + 1)(x - 4)^3$

23. $f(x) = 2x^4 - 5x^3 - 17x^2 + 14x + 41$

24. $f(x) = -3x^4 - 5x^3 + 15x^2 - 5x + 19$

In Exercises 25–28, describe the end behavior of the polynomial function using $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

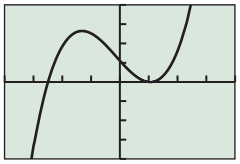
25. $f(x) = 3x^4 - 5x^2 + 3$

26. $f(x) = -x^3 + 7x^2 - 4x + 3$

27. $f(x) = 7x^2 - x^3 + 3x - 4$

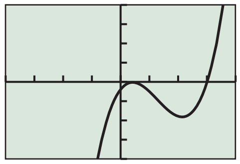
28. $f(x) = x^3 - x^4 + 3x^2 - 2x + 7$

In Exercises 29–32, match the polynomial function with its graph. Approximate all of the real zeros of the function.



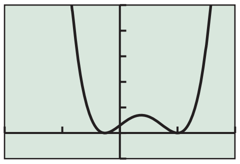
[–4, 4] by [–200, 200]

(a)



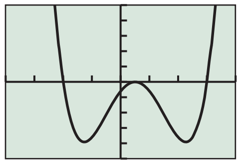
[–4, 4] by [–200, 200]

(b)



[–2, 2] by [–10, 50]

(c)



[–4, 4] by [–50, 50]

(d)

29. $f(x) = 20x^3 + 8x^2 - 83x + 55$

30. $f(x) = 35x^3 - 134x^2 + 93x - 18$

31. $f(x) = 44x^4 - 65x^3 + x^2 + 17x + 3$

32. $f(x) = 4x^4 - 8x^3 - 19x^2 + 23x - 6$

In Exercises 33–38, find the zeros of the function algebraically.

33. $f(x) = x^2 + 2x - 8$ **34.** $f(x) = 3x^2 + 4x - 4$

35. $f(x) = 9x^2 - 3x - 2$ **36.** $f(x) = x^3 - 25x$

37. $f(x) = 3x^3 - x^2 - 2x$ **38.** $f(x) = 5x^3 - 5x^2 - 10x$

In Exercises 39–42, state the degree and list the zeros of the polynomial function. State the multiplicity of each zero and whether the graph crosses the x-axis at the corresponding x-intercept. Then sketch the graph of the polynomial function by hand.

39. $f(x) = x(x - 3)^2$

40. $f(x) = -x^3(x - 2)$

41. $f(x) = (x - 1)^3(x + 2)^2$

42. $f(x) = 7(x - 3)^2(x + 5)^4$

In Exercises 43–48, graph the function in a viewing window that shows all of its x-intercepts and approximate all of its zeros.

43. $f(x) = 2x^3 + 3x^2 - 7x - 6$

44. $f(x) = -x^3 + 3x^2 + 7x - 2$

45. $f(x) = x^3 + 2x^2 - 4x - 7$

46. $f(x) = -x^4 - 3x^3 + 7x^2 + 2x + 8$

47. $f(x) = x^4 + 3x^3 - 9x^2 + 2x + 3$

48. $f(x) = 2x^5 - 11x^4 + 4x^3 + 47x^2 - 42x - 8$

In Exercises 49–52, find the zeros of the function algebraically or graphically.

49. $f(x) = x^3 - 36x$

50. $f(x) = x^3 + 2x^2 - 109x - 110$

51. $f(x) = x^3 - 7x^2 - 49x + 55$

52. $f(x) = x^3 - 4x^2 - 44x + 96$

In Exercises 53–56, using only algebra, find a cubic function with the given zeros. Support by graphing your answer.

53. 3, −4, 6

54. −2, 3, −5

55. $\sqrt{3}, -\sqrt{3}, 4$

56. $1, 1 + \sqrt{2}, 1 - \sqrt{2}$

57. Use cubic regression to fit a curve through the four points given in the table.

x	−3	−1	1	3
y	22	25	12	−5

58. Use cubic regression to fit a curve through the four points given in the table.

x	−2	1	4	7
y	2	5	9	26

59. Use quartic regression to fit a curve through the five points given in the table.

x	3	4	5	6	8
y	−7	−4	−11	8	3

60. Use quartic regression to fit a curve through the five points given in the table.

x	0	4	5	7	13
y	−21	−19	−12	8	3

In Exercises 61–62, explain why the function has at least one real zero.

61. Writing to Learn $f(x) = x^7 + x + 100$

62. Writing to Learn $f(x) = x^9 - x + 50$

63. Stopping Distance A state highway patrol safety division collected the data on stopping distances in Table 2.14 in the next column.

(a) Draw a scatter plot of the data.

(b) Find the quadratic regression model.

(c) Superimpose the regression curve on the scatter plot.

(d) Use the regression model to predict the stopping distance for a vehicle traveling at 25 mph.

(e) Use the regression model to predict the speed of a car if the stopping distance is 300 ft.

Table 2.14 Highway Safety Division

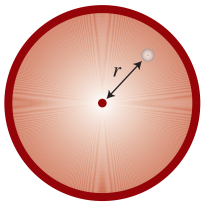
Speed (mph)	Stopping Distance (ft)
10	15.1
20	39.9
30	75.2
40	120.5
50	175.9

64. Analyzing Profit Economists for Smith Brothers, Inc., find the company profit P by using the formula $P = R - C$, where R is the total revenue generated by the business and C is the total cost of operating the business.

(a) Using data from past years, the economists determined that $R(x) = 0.0125x^2 + 412x$ models total revenue, and $C(x) = 12{,}225 + 0.00135x^3$ models the total cost of doing business, where x is the number of customers patronizing the business. How many customers must Smith Bros. have to be profitable each year?

(b) How many customers must there be for Smith Bros. to realize an annual profit of $60,000?

65. Circulation of Blood Research conducted at a national health research project shows that the speed at which a blood cell travels in an artery depends on its distance from the center of the artery. The function $v = 1.19 - 1.87r^2$ models the velocity (in centimeters per second) of a cell that is r centimeters from the center of an artery.



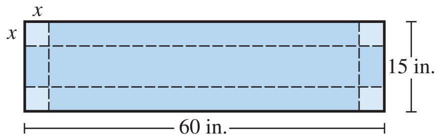
(a) Find a graph of v that reflects values of v appropriate for this problem. Record the viewing-window dimensions.

(b) If a blood cell is traveling at 0.975 cm/sec, estimate the distance the blood cell is from the center of the artery.

66. Volume of a Box Dixie Packaging Co. has contracted to manufacture a box with no top that is to be made by removing squares of width x from the corners of a 15-in. by 60-in. piece of cardboard.

(a) Show that the volume of the box is modeled by $V(x) = x(60 - 2x)(15 - 2x)$.

(b) Determine x so that the volume of the box is at least 450 in.^{3}



67. Volume of a Box Squares of width x are removed from a 10-cm by 25-cm piece of cardboard, and the resulting edges are folded up to form a box with no top. Determine all values of x so that the volume of the resulting box is at most 175 cm^3.

68. Volume of a Box The function $V = 2666x - 210x^2 + 4x^3$ represents the volume of a box that has been made by removing squares of width x from each corner of a rectangular sheet of material and then folding up the sides. What values are possible for x?

Standardized Test Questions

69. True or False The graph of $f(x) = x^3 - x^2 - 2$ crosses the x-axis between $x = 1$ and $x = 2$. Justify your answer.

70. True or False If the graph of $g(x) = (x + a)^2$ is obtained by translating the graph of $f(x) = x^2$ to the right, then a must be positive. Justify your answer.

In Exercises 71–74, solve the problem without using a calculator.

71. Multiple Choice What is the y-intercept of the graph of $f(x) = 2(x - 1)^3 + 5$?

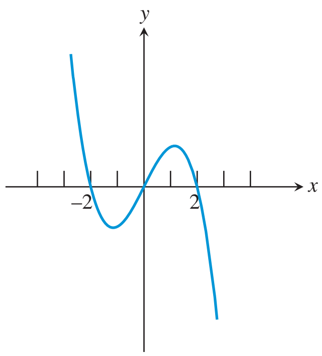
(A) 7 **(B)** 5 **(C)** 3 **(D)** 2 **(E)** 1

72. Multiple Choice What is the multiplicity of the zero $x = 2$ in $f(x) = (x - 2)^2(x + 2)^3(x + 3)^7$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 7

In Exercises 73 and 74, which of the specified functions might have the given graph?

73. Multiple Choice



(A) $f(x) = -x(x + 2)(2 - x)$

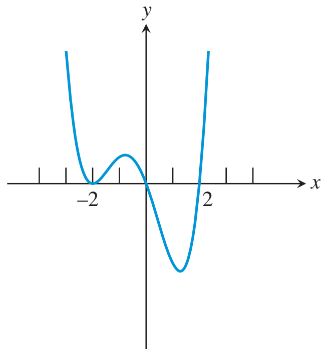
(B) $f(x) = -x(x + 2)(x - 2)$

(C) $f(x) = -x^2(x + 2)(x - 2)$

(D) $f(x) = -x(x + 2)^2(x - 2)$

(E) $f(x) = -x(x + 2)(x - 2)^2$

74. Multiple Choice



(A) $f(x) = x(x + 2)^2(x - 2)$

(B) $f(x) = x(x + 2)^2(2 - x)$

(C) $f(x) = x^2(x + 2)(x - 2)$

(D) $f(x) = x(x + 2)(x - 2)^2$

(E) $f(x) = x^2(x + 2)(x - 2)^2$

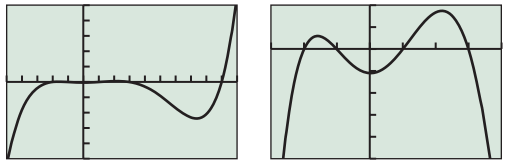
Explorations

In Exercises 75 and 76, two views of the function are given.

75. Writing to Learn Describe why each view of the function

$$f(x) = x^5 - 10x^4 + 2x^3 + 64x^2 - 3x - 55,$$

by itself, may be considered inadequate.



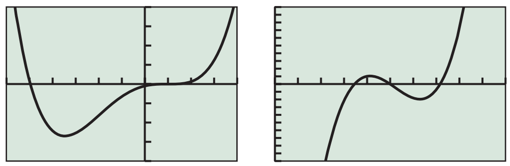
[–5, 10] by [–7500, 7500]
(a)

[–3, 4] by [–250, 100]
(b)

76. Writing to Learn Describe why each view of the function

$$f(x) = 10x^4 + 19x^3 - 121x^2 + 143x - 51,$$

by itself, may be considered inadequate.



[–6, 4] by [–2000, 2000]
(a)

[0.5, 1.5] by [–1, 1]
(b)

In Exercises 77–80, the function has hidden behavior when viewed in the window $[-10, 10]$ by $[-10, 10]$. Describe what behavior is hidden, and state the dimensions of a viewing window that reveals the hidden behavior.

77. $f(x) = 10x^3 - 40x^2 + 50x - 20$

78. $f(x) = 0.5(x^3 - 8x^2 + 12.99x - 5.94)$

79. $f(x) = 11x^3 - 10x^2 + 3x + 5$

80. $f(x) = 33x^3 - 100x^2 + 101x - 40$

Extending the Ideas

81. Graph the left side of the equation

$$3(x^3 - x) = a(x - b)^3 + c.$$

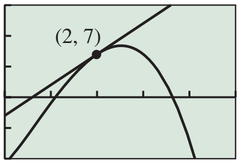
Then explain why there are no real numbers a, b, and c that make the equation true. (*Hint*: Use your knowledge of $y = x^3$ and transformations.)

82. Graph the left side of the equation

$$x^4 + 3x^3 - 2x - 3 = a(x - b)^4 + c.$$

Then explain why there are no real numbers a, b, and c that make the equation true.

83. Looking Ahead to Calculus The figure shows a graph of both $f(x) = -x^3 + 2x^2 + 9x - 11$ and the line L defined by $y = 5(x - 2) + 7$.



[0, 5] by [–10, 15]

(a) Confirm that the point $Q(2, 7)$ is a point of intersection of the two graphs.

(b) Zoom in at point Q to develop a visual understanding that $y = 5(x - 2) + 7$ is a *linear approximation* for $y = f(x)$ near $x = 2$.

(c) Recall that a line is *tangent* to a circle at a point P if it intersects the circle only at point P. View the two graphs in the window $[-5, 5]$ by $[-25, 25]$, and explain why that definition of tangent line is not valid for the graph of f.

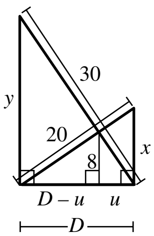
84. Looking Ahead to Calculus Consider the function $f(x) = x^n$ where n is an odd integer.

(a) Suppose that a is a positive number. Show that the slope of the line through the points $P(a, f(a))$ and $Q(-a, f(-a))$ is a^{n-1}.

(b) Let $x_0 = a^{1/(n-1)}$. Find an equation of the line through point $(x_0, f(x_0))$ with the slope a^{n-1}.

(c) Consider the special case $n = 3$ and $a = 3$. Show both the graph of f and the line from part b in the window $[-5, 5]$ by $[-30, 30]$.

85. Derive an Algebraic Model of a Problem Show that the distance x in the figure is a solution of the equation $x^4 - 16x^3 + 500x^2 - 8000x + 32{,}000 = 0$ and find the value of D by following these steps.



(a) Use the similar triangles in the diagram and the properties of proportions learned in geometry to show that

$$\frac{8}{x} = \frac{y - 8}{y}.$$

(b) Show that $y = \dfrac{8x}{x - 8}$.

(c) Show that $y^2 - x^2 = 500$. Then substitute for y, and simplify to obtain the desired degree 4 equation in x.

(d) Find the distance D.

86. Group Learning Activity Consider functions of the form $f(x) = x^3 + bx^2 + x + 1$ where b is a nonzero real number.

(a) Discuss as a group how the value of b affects the graph of the function.

(b) After completing (a), have each member of the group (individually) predict what the graphs of $f(x) = x^3 + 15x^2 + x + 1$ and $g(x) = x^3 - 15x^2 + x + 1$ will look like.

(c) Compare your predictions with each other. Confirm whether they are correct.