What you'll learn about

- Solving Absolute Value Inequalities
- Solving Quadratic Inequalities
- Approximating Solutions to Inequalities
- Projectile Motion

... and why

These techniques are involved in using a graphing utility to solve inequalities in this textbook.

P.7 Solving Inequalities Algebraically and Graphically

Solving Absolute Value Inequalities

The methods for solving inequalities parallel the methods for solving equations. Here are two basic rules we apply to solve absolute value inequalities.

Solving Absolute Value Inequalities

Let u be an algebraic expression in x and let a be a real number with $a \geq 0$.

1. If $|u| < a$, then u is in the interval $(-a, a)$. That is,

$$|u| < a \quad \text{if and only if} \quad -a < u < a.$$

2. If $|u| > a$, then u is in the interval $(-\infty, -a)$ or (a, ∞), that is,

$$|u| > a \quad \text{if and only if} \quad u < -a \text{ or } u > a.$$

The inequalities $<$ and $>$ can be replaced with $\leq$ and $\geq$, respectively. See Figure P.42.

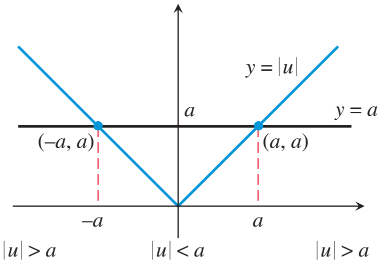


FIGURE P.42 The solution of $|u| < a$ is represented by the portion of the number line where the graph of $y = |u|$ is below the graph of $y = a$. The solution of $|u| > a$ is represented by the portion of the number line where the graph of $y = |u|$ is above the graph of $y = a$.

EXAMPLE 1 Solving an Absolute Value Inequality

Solve $|x - 4| < 8$.

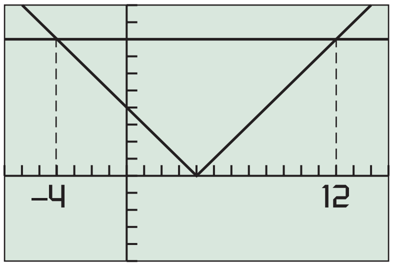
SOLUTION

$$|x - 4| < 8 \qquad \text{Original inequality}$$
$$-8 < x - 4 < 8 \qquad \text{Equivalent double inequality.}$$
$$-4 < x < 12 \qquad \text{Add 4.}$$

As an interval the solution is $(-4, 12)$.

Figure P.43 shows that points on the graph of $y = |x - 4|$ are below the points on the graph of $y = 8$ for values of x between -4 and 12.

Now try Exercise 3.



[–7, 15] by [–5, 10]

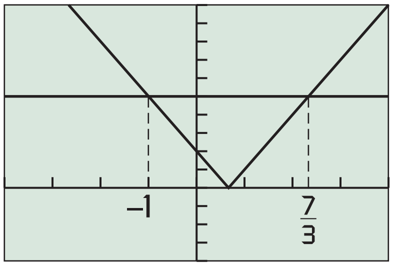
FIGURE P.43 The graphs of $y = |x - 4|$ and $y = 8$. (Example 1)

EXAMPLE 2 Solving Another Absolute Value Inequality

Solve $|3x - 2| \geq 5$.

SOLUTION The solution of this absolute value inequality consists of the solutions of both of these inequalities.

$$3x - 2 \leq -5 \quad \text{or} \quad 3x - 2 \geq 5$$
$$3x \leq -3 \quad \text{or} \quad 3x \geq 7 \qquad \text{Add 2.}$$
$$x \leq -1 \quad \text{or} \quad x \geq \frac{7}{3} \qquad \text{Divide by 3.}$$



[–4, 4] by [–4, 10]

FIGURE P.44 The graphs of $y = |3x - 2|$ and $y = 5$. (Example 2)

The solution consists of all numbers that are in either one of the two intervals $(-\infty, -1]$ or $[7/3, \infty)$, which may be written as $(-\infty, -1] \cup [7/3, \infty)$. The notation "$\cup$" is read as "union."

Figure P.44 shows that points on the graph of $y = |3x - 2|$ are above or on the points on the graph of $y = 5$ for values of x to the left of and including -1 and to the right of and including 7/3. ***Now try Exercise 7.***

Union of Two Sets

The **union of two sets *A* and *B***, denoted by $A \cup B$, is the set of all objects that belong to A or B or both.

Solving Quadratic Inequalities

To solve a quadratic inequality such as $x^2 - x - 12 > 0$ we begin by solving the corresponding quadratic equation $x^2 - x - 12 = 0$. Then we determine the values of x for which the graph of $y = x^2 - x - 12$ lies above the x-axis.

EXAMPLE 3 Solving a Quadratic Inequality

Solve $x^2 - x - 12 > 0$.

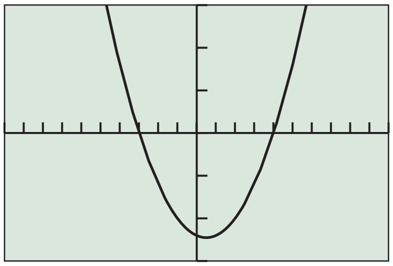
SOLUTION First we solve the corresponding equation $x^2 - x - 12 = 0$.

$$
\begin{aligned}
x^2 - x - 12 &= 0 \\
(x - 4)(x + 3) &= 0 && \text{Factor.} \\
x - 4 = 0 \quad \text{or} \quad x + 3 &= 0 && ab = 0 \Rightarrow a = 0 \text{ or } b = 0 \\
x = 4 \quad \text{or} \quad x &= -3 && \text{Solve for } x.
\end{aligned}
$$

The solutions of the corresponding quadratic equation are -3 and 4, and they are not solutions of the original inequality because $0 > 0$ is false. Figure P.45 shows that the points on the graph of $y = x^2 - x - 12$ are above the x-axis for values of x to the left of -3 and to the right of 4.

The solution of the original inequality is $(-\infty, -3) \cup (4, \infty)$.

Now try Exercise 11.



[–10, 10] by [–15, 15]

FIGURE P.45 The graph of $y = x^2 - x - 12$ appears to cross the x-axis at $x = -3$ and $x = 4$. (Example 3)

In Example 4, the quadratic inequality involves the symbol $\le$. In this case, the solutions of the corresponding quadratic equation are also solutions of the inequality.

EXAMPLE 4 Solving Another Quadratic Inequality

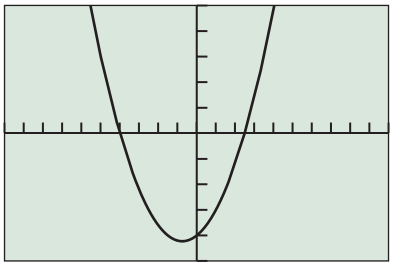
Solve $2x^2 + 3x \le 20$.

SOLUTION First we subtract 20 from both sides of the inequality to obtain $2x^2 + 3x - 20 \le 0$. Next, we solve the corresponding quadratic equation $2x^2 + 3x - 20 = 0$.

$$
\begin{aligned}
2x^2 + 3x - 20 &= 0 \\
(x + 4)(2x - 5) &= 0 && \text{Factor.} \\
x + 4 = 0 \quad \text{or} \quad 2x - 5 &= 0 && ab = 0 \Rightarrow a = 0 \text{ or } b = 0 \\
x = -4 \quad \text{or} \quad x &= \frac{5}{2} && \text{Solve for } x.
\end{aligned}
$$

The solutions of the corresponding quadratic equation are -4 and $5/2 = 2.5$. You can check that they are also solutions of the inequality.

Figure P.46 shows that the points on the graph of $y = 2x^2 + 3x - 20$ are below the x-axis for values of x between -4 and 2.5. The solution of the original inequality is $[-4, 2.5]$. We use square brackets because the numbers -4 and 2.5 are also solutions of the inequality. ***Now try Exercise 9.***



[–10, 10] by [–25, 25]

FIGURE P.46 The graph of $y = 2x^2 + 3x - 20$ appears to be below the x-axis for $-4 < x < 2.5$. (Example 4)

In Examples 3 and 4 the corresponding quadratic equation factored. If this doesn't happen we will need to approximate the zeros of the quadratic equation if it has any. Then we use our accuracy agreement from Section P.5 and write the endpoints of any intervals accurate to two decimal places as illustrated in Example 5.

EXAMPLE 5 Solving a Quadratic Inequality Graphically

Solve $x^2 - 4x + 1 \geq 0$ graphically.

SOLUTION We can use the graph of $y = x^2 - 4x + 1$ in Figure P.47 to determine that the solutions of the equation $x^2 - 4x + 1 = 0$ are about 0.27 and 3.73. Thus, the solution of the original inequality is $(-\infty, 0.27] \cup [3.73, \infty)$. We use square brackets because the zeros of the quadratic equation are solutions of the inequality even though we only have approximations to their values.

Now try Exercise 21.

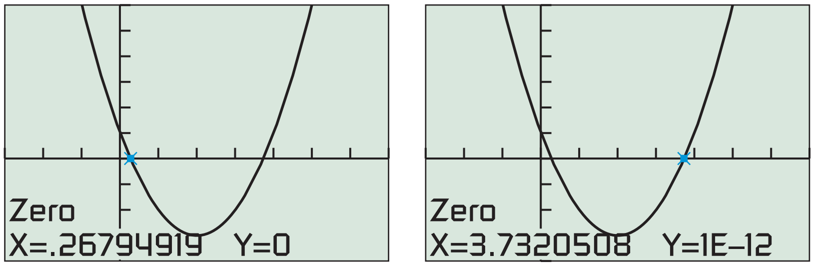


FIGURE P.47 This figure suggests that $y = x^2 - 4x + 1$ is zero for $x \approx 0.27$ and $x \approx 3.73$. (Example 5)

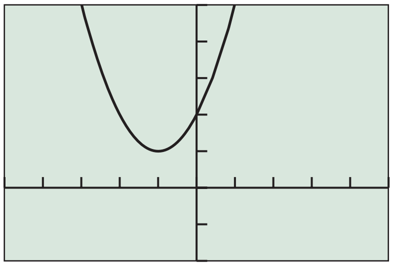


FIGURE P.48 The values of $y = x^2 + 2x + 2$ are never negative. (Example 6)

EXAMPLE 6 Showing That a Quadratic Inequality Has No Solution

Solve $x^2 + 2x + 2 < 0$.

SOLUTION Figure P.48 shows that the graph of $y = x^2 + 2x + 2$ lies above the x-axis for all values for x. Thus, the inequality $x^2 + 2x + 2 < 0$ has *no* solution.

Now try Exercise 25.

Figure P.48 also shows that the solution of the inequality $x^2 + 2x + 2 > 0$ is the set of all real numbers or, in interval notation, $(-\infty, \infty)$. A quadratic inequality can also have exactly one solution (see Exercise 31).

Approximating Solutions to Inequalities

To solve the inequality in Example 7 we approximate the zeros of the corresponding graph. Then we determine the values of x for which the corresponding graph is above or on the x-axis.

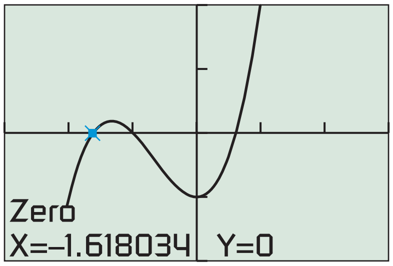
EXAMPLE 7 Solving a Cubic Inequality

Solve $x^3 + 2x^2 - 1 \geq 0$ graphically.

SOLUTION We can use the graph of $y = x^3 + 2x^2 - 1$ in Figure P.49 to show that the solutions of the corresponding equation $x^3 + 2x^2 - 1 = 0$ are approximately -1.62, -1, and 0.62. The points on the graph of $y = x^3 + 2x^2 - 1$ are above the x-axis for values of x between -1.62 and -1, and for values of x to the right of 0.62.

The solution of the inequality is $[-1.62, -1] \cup [0.62, \infty)$. We use square brackets because the zeros of $y = x^3 + 2x^2 - 1$ are also solutions of the inequality.

Now try Exercise 27.



[–3, 3] by [–2, 2]

FIGURE P.49 The graph of $y = x^3 + 2x^2 - 1$ appears to be above the x-axis between the two negative x-intercepts and to the right of the positive x-intercept. (Example 7)

Projectile Motion

The movement of an object that is propelled vertically, but then subject only to the force of gravity, is an example of **projectile motion**.

Projectile Motion

Suppose an object is launched vertically from a point s_0 feet above the ground with an initial velocity of v_0 feet per second. The vertical position s (in feet) of the object t seconds after it is launched is

$$s = -16t^2 + v_0 t + s_0.$$

EXAMPLE 8 Finding Height of a Projectile

A projectile is launched straight up from ground level with an initial velocity of 288 ft/sec.

(a) When will the projectile's height above ground be 1152 ft?

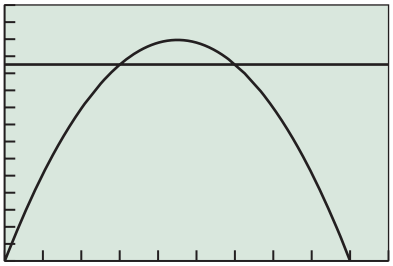
(b) When will the projectile's height above ground be at least 1152 ft?

SOLUTION Here $s_0 = 0$ and $v_0 = 288$. So, the projectile's height is $s = -16t^2 + 288t$.

(a) We need to determine when $s = 1152$.

$$\begin{aligned} s &= -16t^2 + 288t \\ 1152 &= -16t^2 + 288t && \text{Substitute } s = 1152. \\ 16t^2 - 288t + 1152 &= 0 && \text{Add } 16t^2 - 288t. \\ t^2 - 18t + 72 &= 0 && \text{Divide by 16.} \\ (t - 6)(t - 12) &= 0 && \text{Factor.} \\ t = 6 \quad \text{or} \quad t &= 12 && \text{Solve for } t. \end{aligned}$$

The projectile is 1152 ft above ground twice; the first time at $t = 6$ sec on the way up, and the second time at $t = 12$ sec on the way down (Figure P.50).



[0, 20] by [0, 1500]

FIGURE P.50 The graphs of $s = -16t^2 + 288t$ and $s = 1152$. We know from Example 8a that the two graphs intersect at $(6, 1152)$ and $(12, 1152)$.

(b) The projectile will be at least 1152 ft above ground when $s \geq 1152$. We can see from Figure P.50 together with the algebraic work in (a) that the solution is [6, 12]. This means that the projectile is at least 1152 ft above ground for times between $t = 6$ sec and $t = 12$ sec, including 6 and 12 sec.

In Exercise 32 we ask you to use algebra to solve the inequality $s = -16t^2 + 288t \geq 1152$. ***Now try Exercise 33.***

QUICK REVIEW P.7

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–3, solve for x.

1. $-7 < 2x - 3 < 7$
2. $5x - 2 \geq 7x + 4$
3. $|x + 2| = 3$

In Exercises 4–6, factor the expression completely.

4. $4x^2 - 9$
5. $x^3 - 4x$
6. $9x^2 - 16y^2$

In Exercises 7 and 8, reduce the fraction to lowest terms.

7. $\dfrac{z^2 - 25}{z^2 - 5z}$
8. $\dfrac{x^2 + 2x - 35}{x^2 - 10x + 25}$

In Exercises 9 and 10, add the fractions and simplify.

9. $\dfrac{x}{x - 1} + \dfrac{x + 1}{3x - 4}$
10. $\dfrac{2x - 1}{x^2 - x - 2} + \dfrac{x - 3}{x^2 - 3x + 2}$

SECTION P.7 EXERCISES

In Exercises 1–8, solve the inequality algebraically. Write the solution in interval notation and draw its number line graph.

1. $|x + 4| \geq 5$
2. $|2x - 1| > 3.6$
3. $|x - 3| < 2$
4. $|x + 3| \leq 5$
5. $|4 - 3x| - 2 < 4$
6. $|3 - 2x| + 2 > 5$
7. $\left|\dfrac{x + 2}{3}\right| \geq 3$
8. $\left|\dfrac{x - 5}{4}\right| \leq 6$

In Exercises 9–16, solve the inequality. Use algebra to solve the corresponding equation.

9. $2x^2 + 17x + 21 \leq 0$
10. $6x^2 - 13x + 6 \geq 0$
11. $2x^2 + 7x > 15$
12. $4x^2 + 2 < 9x$
13. $2 - 5x - 3x^2 < 0$
14. $21 + 4x - x^2 > 0$
15. $x^3 - x \geq 0$
16. $x^3 - x^2 - 30x \leq 0$

In Exercises 17–26, solve the inequality graphically.

17. $x^2 - 4x < 1$
18. $12x^2 - 25x + 12 \geq 0$
19. $6x^2 - 5x - 4 > 0$
20. $4x^2 - 1 \leq 0$
21. $9x^2 + 12x - 1 \geq 0$
22. $4x^2 - 12x + 7 < 0$
23. $4x^2 + 1 > 4x$
24. $x^2 + 9 \leq 6x$
25. $x^2 - 8x + 16 < 0$
26. $9x^2 + 12x + 4 \geq 0$

In Exercises 27–30, solve the cubic inequality graphically.

27. $3x^3 - 12x + 2 \geq 0$
28. $8x - 2x^3 - 1 < 0$
29. $2x^3 + 2x > 5$
30. $4 \leq 2x^3 + 8x$

31. **Group Activity** Give an example of a quadratic inequality with the indicated solution.
   (a) All real numbers
   (b) No solution
   (c) Exactly one solution
   (d) $[-2, 5]$
   (e) $(-\infty, -1) \cup (4, \infty)$
   (f) $(-\infty, 0] \cup [4, \infty)$

32. **Revisiting Example 8** Solve the inequality $-16t^2 + 288t \geq 1152$ algebraically and compare your answer with the result obtained in Example 10.

33. **Projectile Motion** A projectile is launched straight up from ground level with an initial velocity of 256 ft/sec.
   (a) When will the projectile's height above ground be 768 ft?
   (b) When will the projectile's height above ground be at least 768 ft?
   (c) When will the projectile's height above ground be less than or equal to 768 ft?

34. **Projectile Motion** A projectile is launched straight up from ground level with an initial velocity of 272 ft/sec.
   (a) When will the projectile's height above ground be 960 ft?
   (b) When will the projectile's height above ground be more than 960 ft?
   (c) When will the projectile's height above ground be less than or equal to 960 ft?

35. **Writing to Learn** Explain the role of equation solving in the process of solving an inequality. Give an example.

36. **Travel Planning** Barb wants to drive to a city 105 mi from her home in no more than 2 h. What is the lowest average speed she must maintain on the drive?

37. **Connecting Algebra and Geometry** Consider the collection of all rectangles that have length 2 in. less than twice their width.
   (a) Find the possible widths (in inches) of these rectangles if their perimeters are less than 200 in.
   (b) Find the possible widths (in inches) of these rectangles if their areas are less than or equal to 1200 in.^{2}.

38. **Boyle's Law** For a certain gas, $P = 400/V$, where P is pressure and V is volume. If $20 \leq V \leq 40$, what is the corresponding range for P?

39. **Cash-Flow Planning** A company has current assets (cash, property, inventory, and accounts receivable) of \$200,000 and current liabilities (taxes, loans, and accounts payable) of \$50,000. How much can it borrow if it wants its ratio of assets to liabilities to be no less than 2? Assume the amount borrowed is added to both current assets and current liabilities.

Standardized Test Questions

40. True or False The absolute value inequality $|x - a| < b$, where a and b are real numbers, always has at least one solution. Justify your answer.

41. True or False Every real number is a solution of the absolute value inequality $|x - a| \geq 0$, where a is a real number. Justify your answer.

In Exercises 42–45, solve these problems without using a calculator.

42. Multiple Choice Which of the following is the solution to $|x - 2| < 3$?

(A) $x = -1$ or $x = 5$ **(B)** $[-1, 5)$
(C) $[-1, 5]$ **(D)** $(-\infty, -1) \cup (5, \infty)$
(E) $(-1, 5)$

43. Multiple Choice Which of the following is the solution to $x^2 - 2x + 2 \geq 0$?

(A) $[0, 2]$ **(B)** $(-\infty, 0) \cup (2, \infty)$
(C) $(-\infty, 0] \cup [2, \infty)$ **(D)** All real numbers
(E) There is no solution.

44. Multiple Choice Which of the following is the solution to $x^2 > x$?

(A) $(-\infty, 0) \cup (1, \infty)$ **(B)** $(-\infty, 0] \cup [1, \infty))$
(C) $(1, \infty)$ **(D)** $(0, \infty)$
(E) There is no solution.

45. Multiple Choice Which of the following is the solution to $x^2 \leq 1$?

(A) $(-\infty, 1]$ **(B)** $(-1, 1)$
(C) $[1, \infty)$ **(D)** $[-1, 1]$
(E) There is no solution.

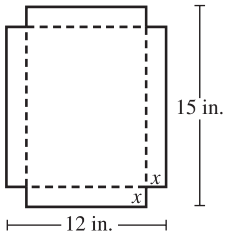
Explorations

46. Constructing a Box with No Top An open box is formed by cutting squares from the corners of a regular piece of cardboard (see figure) and folding up the flaps.

(a) What size corner squares should be cut to yield a box with a volume of 125 in.^{3}?

(b) What size corner squares should be cut to yield a box with a volume more than 125 in.^{3}?

(c) What size corner squares should be cut to yield a box with a volume of at most 125 in.^{3}?



Extending the Ideas

In Exercises 47 and 48, use a combination of algebraic and graphical techniques to solve the inequalities.

47. $|2x^2 + 7x - 15| < 10$ **48.** $|2x^2 + 3x - 20| \geq 10$

CHAPTER P Key Ideas

Properties, Theorems, and Formulas

Procedures