# P.6 Complex Numbers

### What you'll learn about

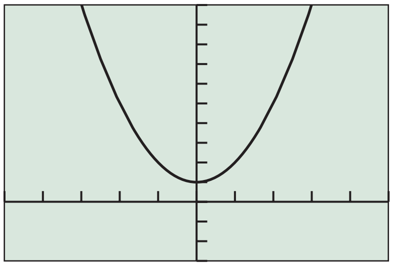
- Complex Numbers
- Operations with Complex Numbers
- Complex Conjugates and Division
- Complex Solutions of Quadratic Equations

### ... and why

The zeros of polynomials are complex numbers.

## Complex Numbers

Figure P.39 shows that the function $f(x) = x^2 + 1$ has no real zeros, so $x^2 + 1 = 0$ has no real-number solutions. To remedy this situation, mathematicians in the 17th century extended the definition of $\sqrt{a}$ to include negative real numbers $a$. First the number $i = \sqrt{-1}$ is defined as a solution of the equation $i^2 + 1 = 0$ and is the **imaginary unit**. Then for any negative real number $\sqrt{a} = \sqrt{|a|} \cdot i$.



[–5, 5] by [–3, 10]

**FIGURE P.39** The graph of $f(x) = x^2 + 1$ has no $x$-intercepts.

### Historical Note

René Descartes (1596–1650) coined the term imaginary in a time when negative solutions to equations were considered *false*. Carl Friedrich Gauss (1777–1855) gave us the term complex number and the symbol $i$ for $\sqrt{-1}$. Today practical applications of complex numbers abound.

The extended system of numbers, called the *complex numbers*, consists of all real numbers and sums of real numbers and real number multiples of $i$. The following are all examples of complex numbers:

$$-6, \quad 5i, \quad \sqrt{5}, \quad -7i, \quad \frac{5}{2}i + \frac{2}{3}, \quad -2 + 3i, \quad 5-3i, \quad \frac{1}{3} + \frac{4}{5}i.$$

**DEFINITION Complex Number**

A **complex number** is any number that can be written in the form

$$a + bi,$$

where $a$ and $b$ are real numbers. The real number $a$ is the **real part**, the real number $b$ is the **imaginary part**, and $a + bi$ is the **standard form**.

A real number $a$ is the complex number $a + 0i$, so *all real numbers are also complex numbers*. If $a = 0$ and $b \neq 0$, then $a + bi$ becomes $bi$, and is an **imaginary number**. For instance, $5i$ and $-7i$ are imaginary numbers.

Two complex numbers are **equal** if and only if their real and imaginary parts are equal. For example,

$$x + yi = 2 + 5i \quad \text{if and only if} \quad x = 2 \text{ and } y = 5.$$

## Operations with Complex Numbers

Adding complex numbers is done by adding their real and imaginary parts separately. Subtracting complex numbers is also done using the same parts.

**DEFINITION Addition and Subtraction of Complex Numbers**

If $a + bi$ and $c + di$ are two complex numbers, then

**Sum:** $(a + bi) + (c + di) = (a + c) + (b + d)i,$

**Difference:** $(a + bi) - (c + di) = (a - c) + (b - d)i.$

### EXAMPLE 1 Adding and Subtracting Complex Numbers

**(a)** $(7 - 3i) + (4 + 5i) = (7 + 4) + (-3 + 5)i = 11 + 2i$

**(b)** $(2 - i) - (8 + 3i) = (2 - 8) + (-1 - 3)i = -6 - 4i$

*Now try Exercise 3.*

The **additive identity** for the complex numbers is $0 = 0 + 0i$. The **additive inverse** of $a + bi$ is $-(a + bi) = -a - bi$ because

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

Many of the properties of real numbers also hold for complex numbers. These include:

- Commutative properties of addition and multiplication,
- Associative properties of addition and multiplication, and
- Distributive properties of multiplication over addition and subtraction.

Using these properties and the fact that $i^2 = -1$, complex numbers can be multiplied by treating them as algebraic expressions.

## EXAMPLE 2 Multiplying Complex Numbers

$$\begin{aligned}(2 + 3i)\cdot(5 - i) &= 2(5 - i) + 3i(5 - i)\\ &= 10 - 2i + 15i - 3i^2\\ &= 10 + 13i - 3(-1)\\ &= 13 + 13i\end{aligned}$$

*Now try Exercise 9.*

We can generalize Example 2 as follows:

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2\\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Many graphers can perform basic calculations on complex numbers. Figure P.40 shows how the operations of Examples 1 and 2 look on some graphers.

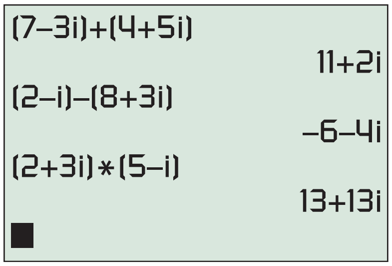



**FIGURE P.40** Complex number operations on a grapher. (Examples 1 and 2)

We compute positive integer powers of complex numbers by treating them as algebraic expressions.

## EXAMPLE 3 Raising a Complex Number to a Power

If $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, find $z^2$ and $z^3$.

**SOLUTION**

$$\begin{aligned}z^2 &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= \frac{1}{4} + \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2\\ &= \frac{1}{4} + \frac{2\sqrt{3}}{4}i + \frac{3}{4}(-1)\\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i\end{aligned}$$

$$\begin{aligned}z^3 = z^2\cdot z &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= -\frac{1}{4} - \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2\\ &= -\frac{1}{4} + 0i + \frac{3}{4}(-1)\\ &= -1\end{aligned}$$

Figure P.41 supports these results numerically. *Now try Exercise 27.*

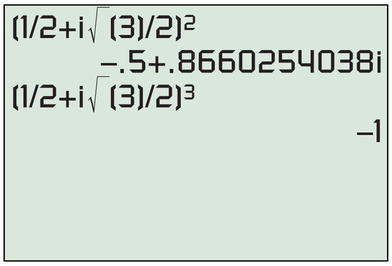



**FIGURE P.41** The square and cube of a complex number. (Example 3)

Example 3 demonstrates that $1/2 + (\sqrt{3}/2)i$ is a cube root of $-1$ and a solution of $x^3 + 1 = 0$. In Section 2.5, complex zeros of polynomial functions will be explored in depth.

## Complex Conjugates and Division

The product of the complex numbers $a + bi$ and $a - bi$ is a positive real number:

$$(a + bi) \cdot (a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

We introduce the following definition to describe this special relationship.

**DEFINITION Complex Conjugate**

The **complex conjugate** of the complex number $z = a + bi$ is

$$\bar{z} = \overline{a + bi} = a - bi.$$

The **multiplicative identity** for the complex numbers is $1 = 1 + 0i$. The **multiplicative inverse**, or **reciprocal**, of $z = a + bi$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

In general, a quotient of two complex numbers, written in fraction form, can be simplified as we just simplified $1/z$—by multiplying the numerator and denominator of the fraction by the complex conjugate of the denominator.

### EXAMPLE 4 Dividing Complex Numbers

Write the complex number in standard form.

**(a)** $\dfrac{2}{3 - i}$ **(b)** $\dfrac{5 + i}{2 - 3i}$

**SOLUTION** Multiply the numerator and denominator by the complex conjugate of the denominator.

**(a)**
$$\frac{2}{3 - i} = \frac{2}{3 - i} \cdot \frac{3 + i}{3 + i} = \frac{6 + 2i}{3^2 + 1^2} = \frac{6}{10} + \frac{2}{10} i = \frac{3}{5} + \frac{1}{5} i$$

**(b)**
$$\frac{5 + i}{2 - 3i} = \frac{5 + i}{2 - 3i} \cdot \frac{2 + 3i}{2 + 3i} = \frac{10 + 15i + 2i + 3i^2}{2^2 + 3^2} = \frac{7 + 17i}{13} = \frac{7}{13} + \frac{17}{13} i$$

*Now try Exercise 33.*

## Complex Solutions of Quadratic Equations

Recall that the solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$, are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The radicand $b^2 - 4ac$ is the **discriminant**, and tells us whether the solutions are real numbers. In particular, if $b^2 - 4ac < 0$, the solutions involve the square root of a negative number and so lead to complex-number solutions. In all, there are three cases, which we now summarize:

### Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$,

- If $b^2 - 4ac > 0$, there are two distinct real solutions.
- If $b^2 - 4ac = 0$, there is one repeated real solution.
- If $b^2 - 4ac < 0$, there is a complex conjugate pair of solutions.

### EXAMPLE 5 Solving a Quadratic Equation

Solve $x^2 + x + 1 = 0$.

**SOLUTION**

**Solve Algebraically**

Using the quadratic formula with $a = b = c = 1$, we obtain

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-(1) \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

So the solutions are $-1/2 + (\sqrt{3}/2)i$ and $-1/2 - (\sqrt{3}/2)i$, a complex conjugate pair.

**Confirm Numerically**

Substituting $-1/2 + (\sqrt{3}/2)i$ into the original equation, we obtain

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1$$
$$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1 = 0.$$

By a similar computation we can confirm the second solution.

*Now try Exercise 41.*

## QUICK REVIEW P.6

In Exercises 1–4, add or subtract, and simplify.

**1.** $(2x + 3) + (-x + 6)$ **2.** $(3y - x) + (2x - y)$

**3.** $(2a + 4d) - (a + 2d)$ **4.** $(6z - 1) - (z + 3)$

In Exercises 5–10, multiply and simplify.

**5.** $(x - 3)(x + 2)$

**6.** $(2x - 1)(x + 3)$

**7.** $(x - \sqrt{2})(x + \sqrt{2})$

**8.** $(x + 2\sqrt{3})(x - 2\sqrt{3})$

**9.** $[x - (1 + \sqrt{2})][x - (1 - \sqrt{2})]$

**10.** $[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})]$

## SECTION P.6 EXERCISES

**Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.**

In Exercises 1–8, write the sum or difference in the standard form $a + bi$ without using a calculator.

**1.** $(2 - 3i) + (6 + 5i)$ **2.** $(2 - 3i) + (3 - 4i)$

**3.** $(7 - 3i) + (6 - i)$ **4.** $(2 + i) - (9i - 3)$

**5.** $(2 - i) + (3 - \sqrt{-3})$

**6.** $(\sqrt{5} - 3i) + (-2 + \sqrt{-9})$

**7.** $(i^2 + 3) - (7 + i^3)$

**8.** $(\sqrt{7} + i^2) - (6 - \sqrt{-81})$

In Exercises 9–16, write the product in standard form without using a calculator.

**9.** $(2 + 3i)(2 - i)$ **10.** $(2 - i)(1 + 3i)$

**11.** $(1 - 4i)(3 - 2i)$ **12.** $(5i - 3)(2i + 1)$

**13.** $(7i - 3)(2 + 6i)$ **14.** $(\sqrt{-4} + i)(6 - 5i)$

**15.** $(-3 - 4i)(1 + 2i)$ **16.** $(\sqrt{-2} + 2i)(6 + 5i)$

In Exercises 17–20, write the expression in the form $bi$, where $b$ is a real number.

**17.** $\sqrt{-16}$ **18.** $\sqrt{-25}$

**19.** $\sqrt{-3}$ **20.** $\sqrt{-5}$

In Exercises 21–24, find the real numbers $x$ and $y$ that make the equation true.

**21.** $2 + 3i = x + yi$ **22.** $3 + yi = x - 7i$

**23.** $(5 - 2i) - 7 = x - (3 + yi)$

**24.** $(x + 6i) = (3 - i) + (4 - 2yi)$

In Exercises 25–28, write the complex number in standard form.

**25.** $(3 + 2i)^2$ **26.** $(1 - i)^3$

**27.** $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^4$ **28.** $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3$

In Exercises 29–32, find the product of the complex number and its conjugate.

**29.** $2 - 3i$ **30.** $5 - 6i$

**31.** $-3 + 4i$ **32.** $-1 - \sqrt{2}i$

In Exercises 33–40, write the expression in standard form without using a calculator.

**33.** $\frac{1}{2 + i}$ **34.** $\frac{i}{2 - i}$

**35.** $\frac{2 + i}{2 - i}$ **36.** $\frac{2 + i}{3i}$

**37.** $\frac{(2 + i)^2(-i)}{1 + i}$ **38.** $\frac{(2 - i)(1 + 2i)}{5 + 2i}$

**39.** $\frac{(1 - i)(2 - i)}{1 - 2i}$ **40.** $\frac{(1 - \sqrt{2}i)(1 + i)}{(1 + \sqrt{2}i)}$

In Exercises 41–44, solve the equation.

**41.** $x^2 + 2x + 5 = 0$ **42.** $3x^2 + x + 2 = 0$

**43.** $4x^2 - 6x + 5 = x + 1$ **44.** $x^2 + x + 11 = 5x - 8$

## Standardized Test Questions

**45. True or False** There are no complex numbers $z$ satisfying $z = -\bar{z}$. Justify your answer.

**46. True or False** For the complex number $i$, $i + i^2 + i^3 + i^4 = 0$. Justify your answer.

In Exercises 47–50, solve the problem without using a calculator.

**47. Multiple Choice** Which of the following is the standard form for the product $(2 + 3i)(2 - 3i)$?

**(A)** $-5 + 12i$ **(B)** $4 - 9i$ **(C)** $13 - 3i$

**(D)** $-5$ **(E)** $13 + 0i$

**48. Multiple Choice** Which of the following is the standard form for the quotient $\frac{1}{i}$?

**(A)** $1$ **(B)** $-1$ **(C)** $i$ **(D)** $-1/i$ **(E)** $0 - i$

**49. Multiple Choice** Assume that $2 - 3i$ is a solution of $ax^2 + bx + c = 0$, where $a, b, c$ are real numbers. Which of the following is also a solution of the equation?

**(A)** $2 + 3i$ **(B)** $-2 - 3i$ **(C)** $-2 + 3i$

**(D)** $3 + 2i$ **(E)** $\frac{1}{2 - 3i}$

**50. Multiple Choice** Which of the following is the standard form for the power $(1 - i)^3$?

**(A)** $-4i$ **(B)** $-2 + 2i$ **(C)** $-2 - 2i$ **(D)** $2 + 2i$ **(E)** $2 - 2i$

## Explorations

**51. Group Activity The Powers of $i$**

**(a)** Simplify the complex numbers $i, i^2, \ldots, i^8$ by evaluating each one.

**(b)** Simplify the complex numbers $i^{-1}, i^{-2}, \ldots, i^{-8}$ by evaluating each one.

**(c)** Evaluate $i^0$.

**(d) Writing to Learn** Discuss your results from (a)–(c) with the members of your group, and write a summary statement about the integer powers of $i$.

**52. Writing to Learn** Describe the nature of the graph of $f(x) = ax^2 + bx + c$ when $a$, $b$, and $c$ are real numbers and the equation $ax^2 + bx + c = 0$ has nonreal complex solutions.

## Extending the Ideas

**53.** Prove that the difference between a complex number and its conjugate is a complex number whose real part is 0.

**54.** Prove that the product of a complex number and its complex conjugate is a complex number whose imaginary part is zero.

**55.** Prove that the complex conjugate of a product of two complex numbers is the product of their complex conjugates.

**56.** Prove that the complex conjugate of a sum of two complex numbers is the sum of their complex conjugates.

**57. Writing to Learn** Explain why $-i$ is a solution of $x^2 - ix + 2 = 0$ but $i$ is not.