P.5 Solving Equations Graphically, Numerically, and Algebraically

What you'll learn about

- Solving Equations Graphically
- Solving Quadratic Equations
- Approximating Solutions of Equations Graphically
- Approximating Solutions of Equations Numerically with Tables
- Solving Equations by Finding Intersections

... and why

These basic techniques are involved in using a graphing utility to solve equations in this textbook.

Solving Equations Graphically

The graph of the equation $y = 2x - 5$ (in x and y) can be used to solve the equation $2x - 5 = 0$ (in x). Using the techniques of Section P.3, we can show algebraically that $x = 5/2$ is a solution of $2x - 5 = 0$. Therefore, the ordered pair $(5/2, 0)$ is a solution of $y = 2x - 5$. Figure P.32 suggests that the x-intercept of the graph of the line $y = 2x - 5$ is the point $(5/2, 0)$ as it should be.

One way to solve an equation graphically is to find all its x-intercepts. There are many graphical techniques that can be used to find x-intercepts.

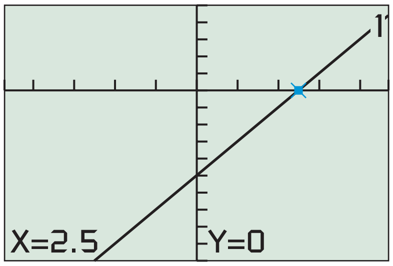

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FIGURE P.32 Using the TRACE feature of a grapher, we see that $(2.5, 0)$ is an x-intercept of the graph of $y = 2x - 5$ and, therefore, $x = 2.5$ is a solution of the equation $2x - 5 = 0$.

EXAMPLE 1 Solving by Finding x-Intercepts

Solve the equation $2x^2 - 3x - 2 = 0$ graphically.

SOLUTION

Solve Graphically

Find the x-intercepts of the graph of $y = 2x^2 - 3x - 2$ (Figure P.33). We use TRACE to see that $(-0.5, 0)$ and $(2, 0)$ are x-intercepts of this graph. Thus, the solutions of this equation are $x = -0.5$ and $x = 2$. Answers obtained graphically are really approximations, although in general they are very good approximations.

Solve Algebraically

In this case, we can use factoring to find exact values.

$$2x^2 - 3x - 2 = 0$$
$$(2x + 1)(x - 2) = 0 \qquad \text{Factor.}$$

We can conclude that

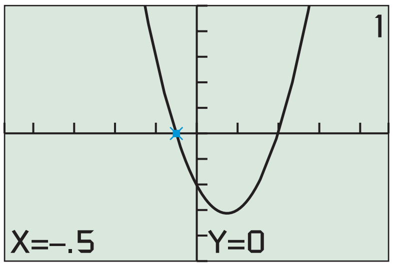
$$2x + 1 = 0 \quad \text{or} \quad x - 2 = 0,$$
$$x = -1/2 \quad \text{or} \quad x = 2.$$

So, $x = -1/2$ and $x = 2$ are the exact solutions of the original equation.

Now try Exercise 1.

Objective

Students will be able to solve equations involving quadratic, absolute value, and fractional expressions by finding x-intercepts or intersections on graphs, by using algebraic techniques, or by using numerical techniques.



[–4.7, 4.7] by [–5, 5]

FIGURE P.33 It appears that $(-0.5, 0)$ and $(2, 0)$ are x-intercepts of the graph of $y = 2x^2 - 3x - 2$. (Example 1)

The algebraic solution procedure used in Example 1 is a special case of the following important property.

Zero Factor Property

Let a and b be real numbers.

$$\text{If } ab = 0, \text{ then } a = 0 \text{ or } b = 0.$$

Solving Quadratic Equations

Linear equations ($ax + b = 0$) and *quadratic equations* are two members of the family of *polynomial equations*, which will be studied in more detail in Chapter 2.

DEFINITION Quadratic Equation in x

A **quadratic equation in x** is one that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$.

We review some of the basic algebraic techniques for solving quadratic equations. One algebraic technique that we have already used in Example 1 is *factoring*.

Quadratic equations of the form $(ax + b)^2 = c$ are fairly easy to solve as illustrated in Example 2.

Square Root Principle

If $t^2 = K > 0$, then $t = \sqrt{K}$ or $t = -\sqrt{K}$.

EXAMPLE 2 Solving by Extracting Square Roots

Solve $(2x - 1)^2 = 9$ algebraically.

SOLUTION

$$\begin{aligned} (2x - 1)^2 &= 9 \\ 2x - 1 &= \pm 3 && \text{Extract square roots.} \\ 2x = 4 \quad &\text{or} \quad 2x = -2 \\ x = 2 \quad &\text{or} \quad x = -1 \end{aligned}$$

Now try Exercise 9.

The technique of Example 2 is more general than you might think because every quadratic equation can be written in the form $(x + b)^2 = c$. The procedure we need to accomplish this is *completing the square*.

Completing the Square

To solve $x^2 + bx = c$ by **completing the square**, add $(b/2)^2$ to both sides of the equation and factor the left side of the new equation.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$
$$\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}$$

To solve a quadratic equation by completing the square, we simply divide both sides of the equation by the coefficient of x^2 and then complete the square as illustrated in Example 3.

EXAMPLE 3 Solving by Completing the Square

Solve $4x^2 - 20x + 17 = 0$ by completing the square.

SOLUTION

$$4x^2 - 20x + 17 = 0$$

$$x^2 - 5x + \frac{17}{4} = 0 \qquad \text{Divide by 4.}$$

$$x^2 - 5x = -\frac{17}{4} \qquad \text{Subtract } \left(\frac{17}{4}\right).$$

Completing the square on the equation above we obtain

$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = -\frac{17}{4} + \left(-\frac{5}{2}\right)^2 \qquad \text{Add } \left(-\frac{5}{2}\right)^2.$$

$$\left(x - \frac{5}{2}\right)^2 = 2 \qquad \text{Factor and simplify.}$$

$$x - \frac{5}{2} = \pm\sqrt{2} \qquad \text{Extract square roots.}$$

$$x = \frac{5}{2} \pm \sqrt{2}$$

$$x = \frac{5}{2} + \sqrt{2} \approx 3.91 \text{ or } x = \frac{5}{2} - \sqrt{2} \approx 1.09$$

Now try Exercise 13.

The procedure of Example 3 can be applied to the general quadratic equation $ax^2 + bx + c = 0$ to produce the following formula for its solutions (see Exercise 68).

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4 Solving Using the Quadratic Formula

Solve the equation $3x^2 - 6x = 5$.

SOLUTION First we subtract 5 from both sides of the equation to put it in the form $ax^2 + bx + c = 0$: $3x^2 - 6x - 5 = 0$. We can see that $a = 3$, $b = -6$, and $c = -5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

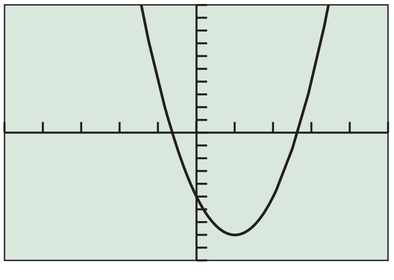
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-5)}}{2(3)} \qquad a = 3,\ b = -6,\ c = -5$$

$$x = \frac{6 \pm \sqrt{96}}{6} \qquad \text{Simplify.}$$

$$x = \frac{6 + \sqrt{96}}{6} \approx 2.63 \quad \text{or} \quad x = \frac{6 - \sqrt{96}}{6} \approx -0.63$$

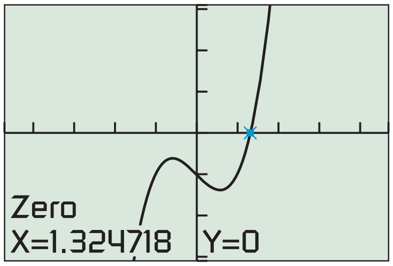
The graph of $y = 3x^2 - 6x - 5$ in Figure P.34 supports that the x-intercepts are approximately -0.63 and 2.63.

Now try Exercise 19.



[–5, 5] by [–10, 10]

FIGURE P.34 The graph of $y = 3x^2 - 6x - 5$. (Example 4)



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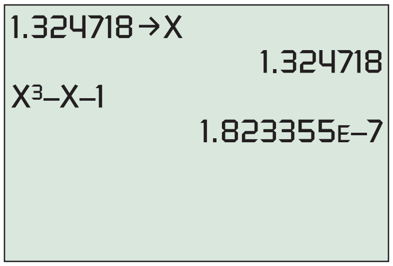
(a)



(b)

FIGURE P.35 The graph of $y = x^3 - x - 1$. (a) shows that (1.324718, 0) is an approximation to the x-intercept of the graph. (b) supports this conclusion. (Example 5)

Solving Quadratic Equations Algebraically

There are four basic ways to solve quadratic equations algebraically.

1. **Factoring** (see Example 1)
2. **Extracting Square Roots** (see Example 2)
3. **Completing the Square** (see Example 3)
4. **Using the Quadratic Formula** (see Example 4)

Approximating Solutions of Equations Graphically

A solution of the equation $x^3 - x - 1 = 0$ is a value of x that makes the value of $y = x^3 - x - 1$ equal to zero. Example 5 illustrates a built-in procedure on graphing calculators to find such values of x.

EXAMPLE 5 Solving Graphically

Solve the equation $x^3 - x - 1 = 0$ graphically.

SOLUTION Figure P.35a suggests that $x = 1.324718$ is the solution we seek. Figure P.35b provides numerical support that $x = 1.324718$ is a close approximation to the solution because, when $x = 1.324718$, $x^3 - x - 1 \approx 1.82 \times 10^{-7}$, which is nearly zero. ***Now try Exercise 31.***

When solving equations graphically, we usually get approximate solutions and not exact solutions. We will use the following agreement about accuracy in this book.

Agreement About Approximate Solutions

For applications, round to a value that is reasonable for the context of the problem. For all others round to two decimal places unless directed otherwise.

With this accuracy agreement, we would report the solution found in Example 5 as 1.32.

Approximating Solutions of Equations Numerically with Tables

The table feature on graphing calculators provides a numerical *zoom-in procedure* that we can use to find accurate solutions of equations. We illustrate this procedure in Example 6 using the same equation of Example 5.

EXAMPLE 6 Solving Using Tables

Solve the equation $x^3 - x - 1 = 0$ using grapher tables.

SOLUTION From Figure P.35a, we know that the solution we seek is between $x = 1$ and $x = 2$. Figure P.36a sets the starting point of the table $(\text{TblStart} = 1)$ at $x = 1$ and increments the numbers in the table $(\Delta\text{Tbl} = 0.1)$ by 0.1. Figure P.36b shows that the zero of $x^3 - x - 1$ is between $x = 1.3$ and $x = 1.4$.

TABLE SETUP
TblStart=1
ΔTbl=.1
Indpnt: **Auto** Ask
Depend: **Auto** Ask

(a)

X	Y_1
1	−1
1.1	−.769
1.2	−.472
1.3	−.103
1.4	.344
1.5	.875
1.6	1.496

$Y_1 \blacksquare X^3 - X - 1$

(b)

FIGURE P.36 (a) gives the setup that produces the table in (b). (Example 6)

The next two steps in this process are shown in Figure P.37.

X	Y_1
1.3	−.103
1.31	−.0619
1.32	−.02
1.33	.02264
1.34	.0661
1.35	.11038
1.36	.15546

$Y_1 \blacksquare X^3 - X - 1$

(a)

X	Y_1
1.32	−.02
1.321	−.0158
1.322	−.0116
1.323	−.0073
1.324	−.0031
1.325	.0012
1.326	.00547

$Y_1 \blacksquare X^3 - X - 1$

(b)

FIGURE P.37 In (a) TblStart = 1.3 and ΔTbl = 0.01, and in (b) TblStart = 1.32 and ΔTbl = 0.001. (Example 6)

From Figure P.37a, we can read that the zero is between $x = 1.32$ and $x = 1.33$; from Figure P.37b, we can read that the zero is between $x = 1.324$ and $x = 1.325$. Because all such numbers round to 1.32, we can report the zero as 1.32 with our accuracy agreement.

Now try Exercise 37.

EXPLORATION 1 Finding Real Zeros of Equations

Consider the equation $4x^2 - 12x + 7 = 0$.

1. Use a graph to show that this equation has two real solutions, one between 0 and 1 and the other between 2 and 3.
2. Use the numerical zoom-in procedure illustrated in Example 6 to find each zero accurate to two decimal places.
3. Use the built-in zero finder (see Example 5) to find the two solutions. Then round them to two decimal places.
4. If you are familiar with the graphical zoom-in process, use it to find each solution accurate to two decimal places.
5. Compare the numbers obtained in parts 2, 3, and 4.
6. Support the results obtained in parts 2, 3, and 4 numerically.
7. Use the numerical zoom-in procedure illustrated in Example 6 to find each zero accurate to six decimal places. Compare with the answer found in part 3 with the zero finder.

Solving Equations by Finding Intersections

Sometimes we can rewrite an equation and solve it graphically by finding the *points of intersection* of two graphs. A point (a, b) is a **point of intersection** of two graphs if it lies on both graphs.

We illustrate this procedure with the absolute value equation in Example 7.

EXAMPLE 7 Solving by Finding Intersections

Solve the equation $|2x - 1| = 6$.

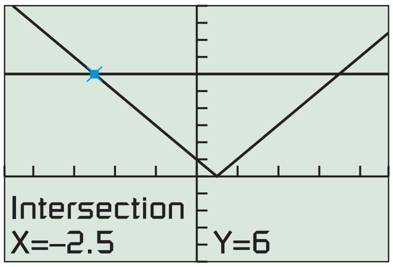
SOLUTION Figure P.38 suggests that the V-shaped graph of $y = |2x - 1|$ intersects the graph of the horizontal line $y = 6$ twice. We can use TRACE or the intersection feature of our grapher to see that the two points of intersection have coordinates $(-2.5, 6)$ and $(3.5, 6)$. This means that the original equation has two solutions: -2.5 and 3.5.

We can use algebra to find the exact solutions. The only two real numbers with absolute value 6 are 6 itself and -6. So, if $|2x - 1| = 6$, then

$$2x - 1 = 6 \quad \text{or} \quad 2x - 1 = -6$$

$$x = \frac{7}{2} = 3.5 \quad \text{or} \quad x = -\frac{5}{2} = -2.5$$

Now try Exercise 39.



[–4.7, 4.7] by [–5, 10]

FIGURE P.38 The graphs of $y = |2x - 1|$ and $y = 6$ intersect at $(-2.5, 6)$ and $(3.5, 6)$. (Example 7)

QUICK REVIEW P.5

In Exercises 1–4, expand the product.

1. $(3x - 4)^2$ **2.** $(2x + 3)^2$

3. $(2x + 1)(3x - 5)$ **4.** $(3y - 1)(5y + 4)$

In Exercises 5–8, factor completely.

5. $25x^2 - 20x + 4$ **6.** $15x^3 - 22x^2 + 8x$

7. $3x^3 + x^2 - 15x - 5$ **8.** $y^4 - 13y^2 + 36$

In Exercises 9 and 10, combine the fractions and reduce the resulting fraction to lowest terms.

9. $\dfrac{x}{2x + 1} - \dfrac{2}{x + 3}$

10. $\dfrac{x + 1}{x^2 - 5x + 6} - \dfrac{3x + 11}{x^2 - x - 6}$

SECTION P.5 EXERCISES

In Exercises 1–6, solve the equation graphically by finding x-intercepts. Confirm by using factoring to solve the equation.

1. $x^2 - x - 20 = 0$ **2.** $2x^2 + 5x - 3 = 0$

3. $4x^2 - 8x + 3 = 0$ **4.** $x^2 - 8x = -15$

5. $x(3x - 7) = 6$ **6.** $x(3x + 11) = 20$

In Exercises 7–12, solve the equation by extracting square roots.

7. $4x^2 = 25$ **8.** $2(x - 5)^2 = 17$

9. $3(x + 4)^2 = 8$ **10.** $4(u + 1)^2 = 18$

11. $2y^2 - 8 = 6 - 2y^2$ **12.** $(2x + 3)^2 = 169$

In Exercises 13–18, solve the equation by completing the square.

13. $x^2 + 6x = 7$ **14.** $x^2 + 5x - 9 = 0$

15. $x^2 - 7x + \frac{5}{4} = 0$ **16.** $4 - 6x = x^2$

17. $2x^2 - 7x + 9 = (x - 3)(x + 1) + 3x$

18. $3x^2 - 6x - 7 = x^2 + 3x - x(x + 1) + 3$

In Exercises 19–24, solve the equation using the quadratic formula.

19. $x^2 + 8x - 2 = 0$ **20.** $2x^2 - 3x + 1 = 0$

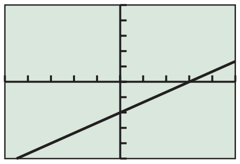
21. $3x + 4 = x^2$ **22.** $x^2 - 5 = \sqrt{3}x$

23. $x(x + 5) = 12$

24. $x^2 - 2x + 6 = 2x^2 - 6x - 26$

In Exercises 25–28, estimate any x- and y-intercepts that are shown in the graph.

25.



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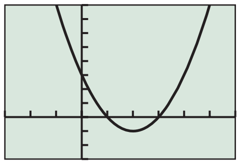
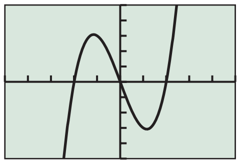
26.



[–3, 6] by [–3, 8]

27.



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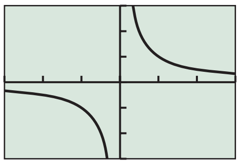
28.



[–3, 3] by [–3, 3]

In Exercises 29–34, solve the equation graphically by finding x-intercepts.

29. $x^2 + x - 1 = 0$ **30.** $4x^2 + 20x + 23 = 0$

31. $x^3 + x^2 + 2x - 3 = 0$ **32.** $x^3 - 4x + 2 = 0$

33. $x^2 + 4 = 4x$ **34.** $x^2 + 2x = -2$

In Exercises 35 and 36, the table permits you to estimate a zero of an expression. State the expression and give the zero as accurately as can be read from the table.

35.

X	Y1
.4	–.04
.41	–.0119
.42	.0164
.43	.0449
.44	.0736
.45	.1025
.46	.1316

Y1 = X²+2X–1

36.

X	Y1
–1.735	–.0177
–1.734	–.0117
–1.733	–.0057
–1.732	3E–4
–1.731	.0063
–1.73	.01228
–1.729	.01826

Y1 = X³–3X

In Exercises 37 and 38, use tables to find the indicated number of solutions of the equation accurate to two decimal places.

37. Two solutions of $x^2 - x - 1 = 0$

38. One solution of $-x^3 + x + 1 = 0$

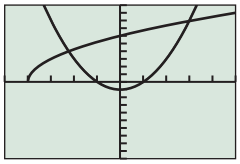
In Exercises 39–44, solve the equation graphically by finding intersections. Confirm your answer algebraically.

39. $|t - 8| = 2$ **40.** $|x + 1| = 4$

41. $|2x + 5| = 7$ **42.** $|3 - 5x| = 4$

43. $|2x - 3| = x^2$ **44.** $|x + 1| = 2x - 3$

45. Interpreting Graphs The graphs in the two viewing windows shown here can be used to solve the equation $3\sqrt{x + 4} = x^2 - 1$ graphically.



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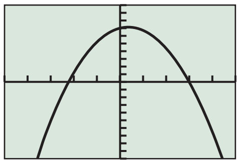
(a)



[–5, 5] by [–10, 10]

(b)

(a) The viewing window in (a) illustrates the intersection method for solving. Identify the two equations that are graphed.

(b) The viewing window in (b) illustrates the x-intercept method for solving. Identify the equation that is graphed.

(c) **Writing to Learn** How are the intersection points in (a) related to the x-intercepts in (b)?

46. Writing to Learn Revisiting Example 6 Explain why all real numbers x that satisfy $1.324 < x < 1.325$ round to 1.32.

In Exercises 47–56, use a method of your choice to solve the equation.

47. $x^2 + x - 2 = 0$ **48.** $x^2 - 3x = 12 - 3(x - 2)$

49. $|2x - 1| = 5$ **50.** $x + 2 - 2\sqrt{x + 3} = 0$

51. $x^3 + 4x^2 - 3x - 2 = 0$ **52.** $x^3 - 4x + 2 = 0$

53. $|x^2 + 4x - 1| = 7$ **54.** $|x + 5| = |x - 3|$

55. $|0.5x + 3| = x^2 - 4$ **56.** $\sqrt{x + 7} = -x^2 + 5$

57. Group Activity Discriminant of a Quadratic The radicand $b^2 - 4ac$ in the quadratic formula is called the **discriminant** of the quadratic polynomial $ax^2 + bx + c$ because it can be used to describe the nature of its zeros.

(a) Writing to Learn If $b^2 - 4ac > 0$, what can you say about the zeros of the quadratic polynomial $ax^2 + bx + c$? Explain your answer.

(b) Writing to Learn If $b^2 - 4ac = 0$, what can you say about the zeros of the quadratic polynomial $ax^2 + bx + c$? Explain your answer.

(c) Writing to Learn If $b^2 - 4ac < 0$, what can you say about the zeros of the quadratic polynomial $ax^2 + bx + c$? Explain your answer.

58. Group Activity Discriminant of a Quadratic Use the information learned in Exercise 57 to create a quadratic polynomial with the following numbers of real zeros. Support your answers graphically.

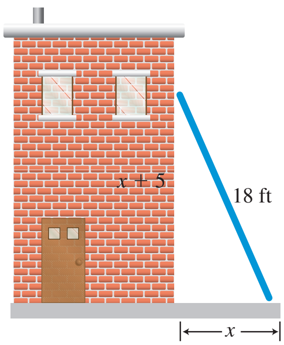
(a) Two real zeros

(b) Exactly one real zero

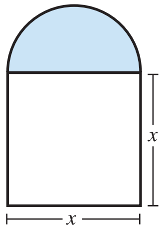
(c) No real zeros

59. Size of a Soccer Field Several of the World Cup '94 soccer matches were played in Stanford University's stadium in Menlo Park, California. The field is 30 yd longer than it is wide, and the area of the field is 8800 yd^2. What are the dimensions of this soccer field?

60. Height of a Ladder John's paint crew knows from experience that its 18-ft ladder is particularly stable when the distance from the ground to the top of the ladder is 5 ft more than the distance from the building to the base of the ladder as shown in the figure. In this position, how far up the building does the ladder reach?



61. Finding the Dimensions of a Norman Window A Norman window has the shape of a square with a semicircle mounted on it. Find the width of the window if the total area of the square and the semicircle is to be 200 ft^2.



Standardized Test Questions

62. True or False If 2 is an x-intercept of the graph of $y = ax^2 + bx + c$, then 2 is a solution of the equation $ax^2 + bx + c = 0$. Justify your answer.

63. True or False If $2x^2 = 18$, then x must be equal to 3. Justify your answer.

In Exercises 64–67, you may use a graphing calculator to solve these problems.

64. Multiple Choice Which of the following are the solutions of the equation $x(x - 3) = 0$?

(A) Only $x = 3$ **(B)** Only $x = -3$

(C) $x = 0$ and $x = -3$ **(D)** $x = 0$ and $x = 3$

(E) There are no solutions.

65. Multiple Choice Which of the following replacements for ? make $x^2 - 5x + ?$ a perfect square?

(A) $-\frac{5}{2}$ **(B)** $\left(-\frac{5}{2}\right)^2$

(C) $(-5)^2$ **(D)** $\left(-\frac{2}{5}\right)^2$

(E) -6

66. Multiple Choice Which of the following are the solutions of the equation $2x^2 - 3x - 1 = 0$?

(A) $\frac{3}{4} \pm \sqrt{17}$ **(B)** $\frac{3 \pm \sqrt{17}}{4}$

(C) $\frac{3 \pm \sqrt{17}}{2}$ **(D)** $\frac{-3 \pm \sqrt{17}}{4}$

(E) $\frac{3 \pm 1}{4}$

67. Multiple Choice Which of the following are the solutions of the equation $|x - 1| = -3$?

(A) Only $x = 4$ **(B)** Only $x = -2$

(C) Only $x = 2$ **(D)** $x = 4$ and $x = -2$

(E) There are no solutions.

Explorations

68. Deriving the Quadratic Formula Follow these steps to use completing the square to solve $ax^2 + bx + c = 0$, $a \neq 0$.

(a) Subtract c from both sides of the original equation and divide both sides of the resulting equation by a to obtain

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

(b) Add the square of one-half of the coefficient of x in (a) to both sides and simplify to obtain

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

(c) Extract square roots in (b) and solve for x to obtain the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Extending the Ideas

69. Finding Number of Solutions Consider the equation $|x^2 - 4| = c$.

(a) Find a value of c for which this equation has four solutions. (There are many such values.)

(b) Find a value of c for which this equation has three solutions. (There is only one such value.)

(c) Find a value of c for which this equation has two solutions. (There are many such values.)

(d) Find a value of c for which this equation has no solutions. (There are many such values.)

(e) **Writing to Learn** Are there any other possible numbers of solutions of this equation? Explain.

70. Sums and Products of Solutions of $ax^2 + bx + c = 0, a \neq 0$ Suppose that $b^2 - 4ac > 0$.

(a) Show that the sum of the two solutions of this equation is $-(b/a)$.

(b) Show that the product of the two solutions of this equation is c/a.

71. Exercise 70 Continued The equation $2x^2 + bx + c = 0$ has two solutions x_1 and x_2. If $x_1 + x_2 = 5$ and $x_1 \cdot x_2 = 3$, find the two solutions.