P.3 Linear Equations and Inequalities

What you'll learn about

- Equations
- Solving Equations
- Linear Equations in One Variable
- Linear Inequalities in One Variable

... and why

These topics provide the foundation for algebraic techniques needed throughout this textbook.

Equations

An **equation** is a statement of equality between two expressions. Here are some properties of equality that we use to solve equations algebraically.

Properties of Equality

Let u, v, w, and z be real numbers, variables, or algebraic expressions.

1. **Reflexive** $u = u$
2. **Symmetric** If $u = v$, then $v = u$.
3. **Transitive** If $u = v$, and $v = w$, then $u = w$.
4. **Addition** If $u = v$ and $w = z$, then $u + w = v + z$.
5. **Multiplication** If $u = v$ and $w = z$, then $uw = vz$.

Solving Equations

A **solution of an equation in *x*** is a value of x for which the equation is true. To **solve an equation in *x*** means to find all values of x for which the equation is true, that is, to find all solutions of the equation.

EXAMPLE 1 Confirming a Solution

Prove that $x = -2$ is a solution of the equation $x^3 - x + 6 = 0$.

SOLUTION

$$(-2)^3 - (-2) + 6 \stackrel{?}{=} 0$$
$$-8 + 2 + 6 \stackrel{?}{=} 0$$
$$0 = 0$$

Now try Exercise 1.

Linear Equations in One Variable

The most basic equation in algebra is a *linear equation*.

DEFINITION Linear Equation in x

A **linear equation in *x*** is one that can be written in the form

$$ax + b = 0,$$

where a and b are real numbers with $a \neq 0$.

The equation $2z - 4 = 0$ is linear in the variable z. The equation $3u^2 - 12 = 0$ is *not* linear in the variable u. A linear equation in one variable has exactly one solution. We solve such an equation by transforming it into an *equivalent equation* whose solution is obvious. Two or more equations are **equivalent** if they have the same solutions. For example, the equations $2z - 4 = 0$, $2z = 4$, and $z = 2$ are all equivalent. Here are operations that produce equivalent equations.

Operations for Equivalent Equations

An equivalent equation is obtained if one or more of the following operations are performed.

Operation	Given Equation	Equivalent Equation
1. Combine like terms, reduce fractions, and remove grouping symbols.	$2x + x = \frac{3}{9}$	$3x = \frac{1}{3}$
2. Perform the same operation on both sides.		
(a) Add (-3).	$x + 3 = 7$	$x = 4$
(b) Subtract $(2x)$.	$5x = 2x + 4$	$3x = 4$
(c) Multiply by a nonzero constant $(1/3)$.	$3x = 12$	$x = 4$
(d) Divide by a nonzero constant (3).	$3x = 12$	$x = 4$

The next two examples illustrate how to use equivalent equations to solve linear equations.

EXAMPLE 2 Solving a Linear Equation

Solve $2(2x - 3) + 3(x + 1) = 5x + 2$. Support the result with a calculator.

SOLUTION

$$
\begin{aligned}
2(2x - 3) + 3(x + 1) &= 5x + 2 \\
4x - 6 + 3x + 3 &= 5x + 2 && \text{Distributive property} \\
7x - 3 &= 5x + 2 && \text{Combine like terms.} \\
2x &= 5 && \text{Add 3, and subtract } 5x. \\
x &= 2.5 && \text{Divide by 2.}
\end{aligned}
$$

To support our algebraic work we can use a calculator to evaluate the original equation for $x = 2.5$. Figure P.17 shows that each side of the original equation is equal to 14.5 if $x = 2.5$.

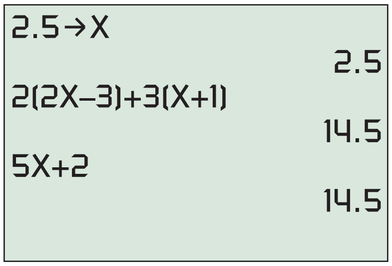


FIGURE P.17 The top line stores the number 2.5 into the variable x. (Example 2)

Now try Exercise 23.

If an equation involves fractions, find the least common denominator (LCD) of the fractions and multiply both sides by the LCD. This is sometimes referred to as *clearing the equation of fractions*. Example 3 illustrates.

Integers and Fractions

Notice in Example 3 that $2 = \frac{2}{1}$.

EXAMPLE 3 Solving a Linear Equation Involving Fractions

Solve

$$\frac{5y - 2}{8} = 2 + \frac{y}{4}.$$

SOLUTION The denominators are 8, 1, and 4. The LCD of the fractions is 8. (See Appendix A.3 if necessary.)

$$\frac{5y - 2}{8} = 2 + \frac{y}{4}$$

$$8\left(\frac{5y - 2}{8}\right) = 8\left(2 + \frac{y}{4}\right) \quad \text{Multiply by the LCD 8.}$$

$$8 \cdot \frac{5y - 2}{8} = 8 \cdot 2 + 8 \cdot \frac{y}{4} \quad \text{Distributive property}$$

$$5y - 2 = 16 + 2y \quad \text{Simplify.}$$

$$5y = 18 + 2y \quad \text{Add 2.}$$

$$3y = 18 \quad \text{Subtract } 2y.$$

$$y = 6 \quad \text{Divide by 3.}$$

We leave it to you to check the solution using either paper and pencil or a calculator.

Now try Exercise 25.

Linear Inequalities in One Variable

We used inequalities to describe order on the number line in Section P.1. For example, if x is to the left of 2 on the number line, or if x is any real number less than 2, we write $x < 2$. The most basic inequality in algebra is a *linear inequality*.

DEFINITION Linear Inequality in x

A **linear inequality in x** is one that can be written in the form

$$ax + b < 0, \quad ax + b \le 0, \quad ax + b > 0, \quad \text{or} \quad ax + b \ge 0,$$

where a and b are real numbers with $a \ne 0$.

To **solve an inequality in x** means to find all values of x for which the inequality is true. A **solution of an inequality in x** is a value of x for which the inequality is true. The set of all solutions of an inequality is the **solution set** of the inequality. We **solve an inequality** by finding its solution set. Here is a list of properties we use to solve inequalities.

Direction of an Inequality

Multiplying (or dividing) an inequality by a positive number preserves the direction of the inequality. Multiplying (or dividing) an inequality by a negative number reverses the direction.

Properties of Inequalities

Let u, v, w, and z be real numbers, variables, or algebraic expressions, and c a real number.

1. Transitive If $u < v$ and $v < w$, then $u < w$.

2. Addition If $u < v$, then $u + w < v + w$.
If $u < v$ and $w < z$, then $u + w < v + z$.

3. Multiplication If $u < v$ and $c > 0$, then $uc < vc$.
If $u < v$ and $c < 0$, then $uc > vc$.

The above properties are true if $<$ is replaced by $\le$. There are similar properties for $>$ and $\ge$.

The set of solutions of a linear inequality in one variable forms an interval of real numbers. Just as with linear equations, we solve a linear inequality by transforming it into an *equivalent inequality* whose solutions are obvious. Two or more inequalities are **equivalent** if they have the same set of solutions. The properties of inequalities listed above describe operations that transform an inequality into an equivalent one.

EXAMPLE 4 Solving a Linear Inequality

Solve $3(x - 1) + 2 \leq 5x + 6$.

SOLUTION

$$
\begin{aligned}
3(x - 1) + 2 &\leq 5x + 6 && \\
3x - 3 + 2 &\leq 5x + 6 && \text{Distributive property} \\
3x - 1 &\leq 5x + 6 && \text{Simplify.} \\
3x &\leq 5x + 7 && \text{Add 1.} \\
-2x &\leq 7 && \text{Subtract } 5x. \\
\left(-\frac{1}{2}\right) \cdot -2x &\geq \left(-\frac{1}{2}\right) \cdot 7 && \text{Multiply by } -1/2. \text{ (The inequality reverses.)} \\
x &\geq -3.5 &&
\end{aligned}
$$

The solution set of the inequality is the set of all real numbers greater than or equal to -3.5. In interval notation, the solution set is $[-3.5, \infty)$.

Now try Exercise 41.

Because the solution set of a linear inequality is an interval of real numbers, we can display the solution set with a number line graph as illustrated in Example 5.

EXAMPLE 5 Solving a Linear Inequality Involving Fractions

Solve the inequality and graph its solution set.

$$\frac{x}{3} + \frac{1}{2} > \frac{x}{4} + \frac{1}{3}$$

SOLUTION The LCD of the fractions is 12.

$$
\begin{aligned}
\frac{x}{3} + \frac{1}{2} &> \frac{x}{4} + \frac{1}{3} && \text{The original inequality} \\
12 \cdot \left(\frac{x}{3} + \frac{1}{2}\right) &> 12 \cdot \left(\frac{x}{4} + \frac{1}{3}\right) && \text{Multiply by the LCD 12.} \\
4x + 6 &> 3x + 4 && \text{Simplify.} \\
x + 6 &> 4 && \text{Subtract } 3x. \\
x &> -2 && \text{Subtract 6.}
\end{aligned}
$$

The solution set is the interval $(-2, \infty)$. Its graph is shown in Figure P.18.

FIGURE P.18 The graph of the solution set of the inequality in Example 5.

Now try Exercise 43.

Sometimes two inequalities are combined in a **double inequality**, whose solution set is a double inequality with x isolated as the middle term. Example 6 illustrates.

EXAMPLE 6 Solving a Double Inequality

Solve the inequality and graph its solution set.

$$-3 < \frac{2x+5}{3} \le 5$$

SOLUTION

$$-3 < \frac{2x+5}{3} \le 5$$

$$-9 < 2x + 5 \le 15 \quad \text{Multiply by 3.}$$

$$-14 < 2x \le 10 \quad \text{Subtract 5.}$$

$$-7 < x \le 5 \quad \text{Divide by 2.}$$

The solution set is the set of all real numbers greater than -7 and less than or equal to 5. In interval notation, the solution is set $(-7, 5]$. Its graph is shown in Figure P.19.

Now try Exercise 47.

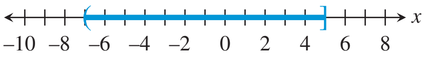


FIGURE P.19 The graph of the solution set of the double inequality in Example 6.

QUICK REVIEW P.3

In Exercises 1 and 2, simplify the expression by combining like terms.

1. $2x + 5x + 7 + y - 3x + 4y + 2$

2. $4 + 2x - 3z + 5y - x + 2y - z - 2$

In Exercises 3 and 4, use the distributive property to expand the products. Simplify the resulting expression by combining like terms.

3. $3(2x - y) + 4(y - x) + x + y$

4. $5(2x + y - 1) + 4(y - 3x + 2) + 1$

In Exercises 5–10, use the LCD to combine the fractions. Simplify the resulting fraction.

5. $\frac{2}{y} + \frac{3}{y}$ **6.** $\frac{1}{y-1} + \frac{3}{y-2}$

7. $2 + \frac{1}{x}$ **8.** $\frac{1}{x} + \frac{1}{y} - x$

9. $\frac{x+4}{2} + \frac{3x-1}{5}$ **10.** $\frac{x}{3} + \frac{x}{4}$

SECTION P.3 EXERCISES

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, find which values of x are solutions of the equation.

1. $2x^2 + 5x = 3$

(a) $x = -3$ **(b)** $x = -\frac{1}{2}$ **(c)** $x = \frac{1}{2}$

2. $\frac{x}{2} + \frac{1}{6} = \frac{x}{3}$

(a) $x = -1$ **(b)** $x = 0$ **(c)** $x = 1$

3. $\sqrt{1 - x^2} + 2 = 3$

(a) $x = -2$ **(b)** $x = 0$ **(c)** $x = 2$

4. $(x - 2)^{1/3} = 2$

(a) $x = -6$ **(b)** $x = 8$ **(c)** $x = 10$

In Exercises 5–10, determine whether the equation is linear in x.

5. $5 - 3x = 0$ **6.** $5 = 10/2$

7. $x + 3 = x - 5$ **8.** $x - 3 = x^2$

9. $2\sqrt{x} + 5 = 10$ **10.** $x + \frac{1}{x} = 1$

In Exercises 11–24, solve the equation without using a calculator.

11. $3x = 24$ **12.** $4x = -16$

13. $3t - 4 = 8$ **14.** $2t - 9 = 3$

15. $2x - 3 = 4x - 5$ **16.** $4 - 2x = 3x - 6$

17. $4 - 3y = 2(y + 4)$ **18.** $4(y - 2) = 5y$

19. $\frac{1}{2}x = \frac{7}{8}$ **20.** $\frac{2}{3}x = \frac{4}{5}$

21. $\frac{1}{2}x + \frac{1}{3} = 1$

22. $\frac{1}{3}x + \frac{1}{4} = 1$

23. $2(3 - 4z) - 5(2z + 3) = z - 17$

24. $3(5z - 3) - 4(2z + 1) = 5z - 2$

In Exercises 25–28, solve the equation. Support your answer with a calculator.

25. $\frac{2x - 3}{4} + 5 = 3x$

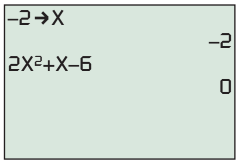
26. $2x - 4 = \frac{4x - 5}{3}$

27. $\frac{t + 5}{8} - \frac{t - 2}{2} = \frac{1}{3}$

28. $\frac{t - 1}{3} + \frac{t + 5}{4} = \frac{1}{2}$

29. Writing to Learn Write a statement about solutions of equations suggested by the computations in the figure.

(a)



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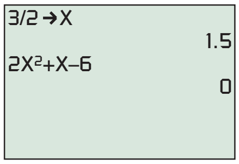
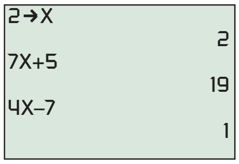
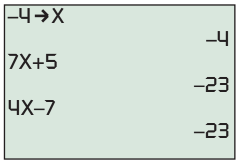


30. Writing to Learn Write a statement about solutions of equations suggested by the computations in the figure.

(a)



(b)



In Exercises 31–34, find which values of x are solutions of the inequality.

31. $2x - 3 < 7$

(a) $x = 0$ **(b)** $x = 5$ **(c)** $x = 6$

32. $3x - 4 \geq 5$

(a) $x = 0$ **(b)** $x = 3$ **(c)** $x = 4$

33. $-1 < 4x - 1 \leq 11$

(a) $x = 0$ **(b)** $x = 2$ **(c)** $x = 3$

34. $-3 \leq 1 - 2x \leq 3$

(a) $x = -1$ **(b)** $x = 0$ **(c)** $x = 2$

In Exercises 35–42, solve the inequality, and draw a number line graph of the solution set.

35. $x - 4 < 2$

36. $x + 3 > 5$

37. $2x - 1 \leq 4x + 3$

38. $3x - 1 \geq 6x + 8$

39. $2 \leq x + 6 < 9$

40. $-1 \leq 3x - 2 < 7$

41. $2(5 - 3x) + 3(2x - 1) \leq 2x + 1$

42. $4(1 - x) + 5(1 + x) > 3x - 1$

In Exercises 43–54, solve the inequality.

43. $\frac{5x + 7}{4} \leq -3$

44. $\frac{3x - 2}{5} > -1$

45. $4 \geq \frac{2y - 5}{3} \geq -2$

46. $1 > \frac{3y - 1}{4} > -1$

47. $0 \leq 2z + 5 < 8$

48. $-6 < 5t - 1 < 0$

49. $\frac{x - 5}{4} + \frac{3 - 2x}{3} < -2$

50. $\frac{3 - x}{2} + \frac{5x - 2}{3} < -1$

51. $\frac{2y - 3}{2} + \frac{3y - 1}{5} < y - 1$

52. $\frac{3 - 4y}{6} - \frac{2y - 3}{8} \geq 2 - y$

53. $\frac{1}{2}(x - 4) - 2x \leq 5(3 - x)$

54. $\frac{1}{2}(x + 3) + 2(x - 4) < \frac{1}{3}(x - 3)$

In Exercises 55–58, find the solutions of the equation or inequality displayed in Figure P.20.

55. $x^2 - 2x < 0$

56. $x^2 - 2x = 0$

57. $x^2 - 2x > 0$

58. $x^2 - 2x \leq 0$

X	Y_1	
0	0	
1	–1	
2	0	
3	3	
4	8	
5	15	
6	24	
Y_1 = X²–2X		

FIGURE P.20 The second column gives values of $y_1 = x^2 - 2x$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.

59. Writing to Learn Explain how the second equation was obtained from the first.

$$x - 3 = 2x + 3, \quad 2x - 6 = 4x + 6$$

60. Writing to Learn Explain how the second equation was obtained from the first.

$$2x - 1 = 2x - 4, \quad x - \frac{1}{2} = x - 2$$

61. Group Activity Determine whether the two equations are equivalent.

(a) $3x = 6x + 9, \quad x = 2x + 9$

(b) $6x + 2 = 4x + 10, \quad 3x + 1 = 2x + 5$

62. Group Activity Determine whether the two equations are equivalent.

(a) $3x + 2 = 5x - 7, \quad -2x + 2 = -7$

(b) $2x + 5 = x - 7, \quad 2x = x - 7$

Standardized Test Questions

63. True or False $-6 > -2$. Justify your answer.

64. True or False $2 \le \frac{6}{3}$. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve these problems.

65. Multiple Choice Which of the following equations is equivalent to the equation $3x + 5 = 2x + 1$?

(A) $3x = 2x$
(B) $3x = 2x + 4$
(C) $\frac{3}{2}x + \frac{5}{2} = x + 1$
(D) $3x + 6 = 2x$
(E) $3x = 2x - 4$

66. Multiple Choice Which of the following inequalities is equivalent to the inequality $-3x < 6$?

(A) $3x < -6$
(B) $x < 10$
(C) $x > -2$
(D) $x > 2$
(E) $x > 3$

67. Multiple Choice Which of the following is the solution to the equation $x(x + 1) = 0$?

(A) $x = 0$ or $x = -1$
(B) $x = 0$ or $x = 1$
(C) Only $x = -1$
(D) Only $x = 0$
(E) Only $x = 1$

68. Multiple Choice Which of the following represents an equation equivalent to the equation

$$\frac{2x}{3} + \frac{1}{2} = \frac{x}{4} - \frac{1}{3}$$

that is cleared of fractions?

(A) $2x + 1 = x - 1$
(B) $8x + 6 = 3x - 4$
(C) $4x + 3 = \frac{3}{2}x - 2$
(D) $4x + 3 = 3x - 4$
(E) $4x + 6 = 3x - 4$

Explorations

69. Testing Inequalities on a Calculator

(a) The calculator we use indicates that the statement $2 < 3$ is true by returning the value 1 (for true) when $2 < 3$ is entered. Try it with your calculator.

(b) The calculator we use indicates that the statement $2 < 1$ is false by returning the value 0 (for false) when $2 < 1$ is entered. Try it with your calculator.

(c) Use your calculator to test which of these two numbers is larger: 799/800, 800/801.

(d) Use your calculator to test which of these two numbers is larger: $-102/101$, $-103/102$.

(e) If your calculator returns 0 when you enter $2x + 1 < 4$, what can you conclude about the value stored in x?

Extending the Ideas

70. Perimeter of a Rectangle The formula for the perimeter P of a rectangle is

$$P = 2(L + W).$$

Solve this equation for W.

71. Area of a Trapezoid The formula for the area A of a trapezoid is

$$A = \frac{1}{2}h(b_1 + b_2).$$

Solve this equation for b_1.

72. Volume of a Sphere The formula for the volume V of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

Solve this equation for r.

73. Celsius and Fahrenheit The formula for Celsius temperature in terms of Fahrenheit temperature is

$$C = \frac{5}{9}(F - 32).$$

Solve the equation for F.