P.2 Cartesian Coordinate System

What you'll learn about

- Cartesian Plane
- Absolute Value of a Real Number
- Distance Formulas
- Midpoint Formulas
- Equations of Circles
- Applications

... and why

These topics provide the foundation for the material that will be covered in this textbook.

Cartesian Plane

The points in a plane correspond to ordered pairs of real numbers, just as the points on a line can be associated with individual real numbers. This correspondence creates the **Cartesian plane**, or the **rectangular coordinate system** in the plane.

To construct a rectangular coordinate system, or a Cartesian plane, draw a pair of perpendicular real number lines, one horizontal and the other vertical, with the lines intersecting at their respective 0-points (Figure P.6). The horizontal line is usually the **x-axis** and the vertical line is usually the **y-axis**. The positive direction on the x-axis is to the right, and the positive direction on the y-axis is up. Their point of intersection, O, is the **origin of the Cartesian plane**.

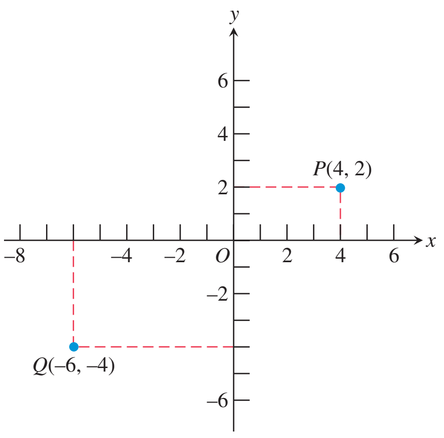


FIGURE P.6 The Cartesian coordinate plane.

Each point P of the plane is associated with an **ordered pair (x, y)** of real numbers, the **(Cartesian) coordinates of the point**. The **x-coordinate** represents the intersection of the x-axis with the perpendicular from P, and the **y-coordinate** represents the intersection of the y-axis with the perpendicular from P. Figure P.6 shows the points P and Q with coordinates $(4, 2)$ and $(-6, -4)$, respectively. As with real numbers and a number line, we use the ordered pair (a, b) for both the name of the point and its coordinates.

The coordinate axes divide the Cartesian plane into four **quadrants**, as shown in Figure P.7.

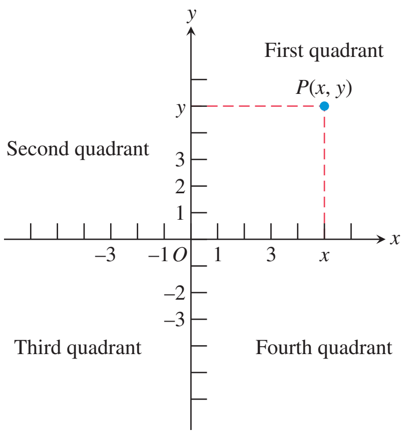


FIGURE P.7 The four quadrants. Points on the x- or y-axis are not in any quadrant.

EXAMPLE 1 Plotting Data on U.S. Exports to Mexico

The value in billions of dollars of U.S. exports to Mexico from 2000 to 2007 is given in Table P.2. Plot the (year, export value) ordered pairs in a rectangular coordinate system.

Table P.2 U.S. Exports to Mexico

Year	U.S. Exports (billions of dollars)
2000	111.3
2001	101.3
2002	97.5
2003	97.4
2004	110.8
2005	120.4
2006	134.0
2007	136.0

Source: U.S. Census Bureau, The World Almanac and Book of Facts 2009.

SOLUTION The points are plotted in Figure P.8 on page 13.

Now try Exercise 31.

A **scatter plot** is a plotting of the (x, y) data pairs on a Cartesian plane. Figure P.8 shows a scatter plot of the data from Table P.2.

Absolute Value of a Real Number

The *absolute value of a real number* suggests its **magnitude** (size). For example, the absolute value of 3 is 3 and the absolute value of -5 is 5.

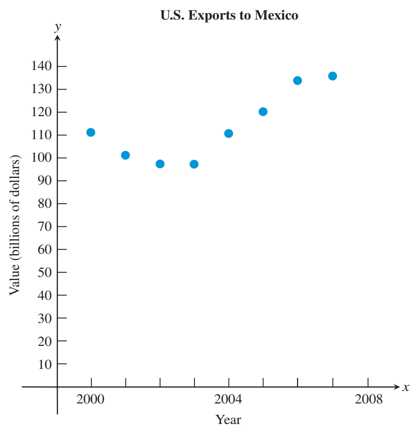


FIGURE P.8 The graph for Example 1.

DEFINITION Absolute Value of a Real Number

The **absolute value of a real number *a*** is

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ -a, & \text{if } a < 0 \\ 0, & \text{if } a = 0. \end{cases}$$

EXAMPLE 2 Using the Definition of Absolute Value

Evaluate:

(a) $|-4|$ **(b)** $|\pi - 6|$

SOLUTION

(a) Because $-4 < 0$, $|-4| = -(-4) = 4$.

(b) Because $\pi \approx 3.14$, $\pi - 6$ is negative, so $\pi - 6 < 0$. Thus,
$|\pi - 6| = -(\pi - 6) = 6 - \pi \approx 2.858$. ***Now try Exercise 9.***

Here is a summary of some important properties of absolute value.

Properties of Absolute Value

Let a and b be real numbers.

1. $|a| \geq 0$ **2.** $|-a| = |a|$

3. $|ab| = |a||b|$ **4.** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

Distance Formulas

The *distance* between -1 and 4 on the number line is 5 (see Figure P.9). This distance may be found by subtracting the smaller number from the larger: $4 - (-1) = 5$. If we use absolute value, the order of subtraction does not matter: $|4 - (-1)| = |-1 - 4| = 5$.

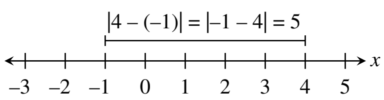


FIGURE P.9 Finding the distance between -1 and 4.

Absolute Value and Distance

If we let $b = 0$ in the distance formula, we see that the distance between a and 0 is $|a|$. Thus, the absolute value of a number is its distance from zero.

Distance Formula (Number Line)

Let a and b be real numbers. The **distance between *a* and *b*** is

$$|a - b|.$$

Note that $|a - b| = |b - a|$.

To find the *distance* between two points that lie on the same horizontal or vertical line in the Cartesian plane, we use the distance formula for points on a number line. For example, the distance between points x_1 and x_2 on the x-axis is $|x_1 - x_2| = |x_2 - x_1|$ and the distance between points y_1 and y_2 on the y-axis is $|y_1 - y_2| = |y_2 - y_1|$.

To find the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ that do not lie on the same horizontal or vertical line, we form the right triangle determined by P, Q, and $R(x_2, y_1)$, (Figure P.10).

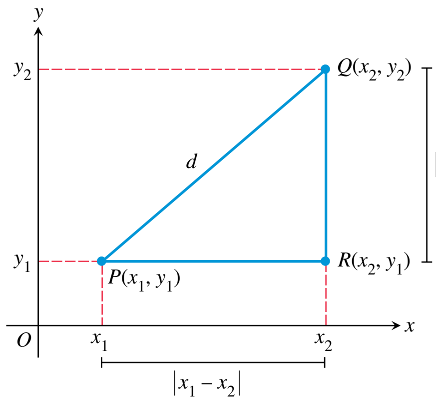


FIGURE P.10 Forming a right triangle with hypotenuse $\overline{PQ}$.

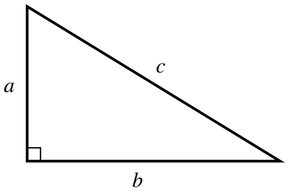


FIGURE P.11 The Pythagorean Theorem: $c^2 = a^2 + b^2$.

The distance from P to R is $|x_1 - x_2|$, and the distance from R to Q is $|y_1 - y_2|$. By the **Pythagorean Theorem** (see Figure P.11), the distance d between P and Q is

$$d = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$

Because $|x_1 - x_2|^2 = (x_1 - x_2)^2$ and $|y_1 - y_2|^2 = (y_1 - y_2)^2$, we obtain the following formula.

Distance Formula (Coordinate Plane)

The **distance d between points $P(x_1, y_1)$ and $Q(x_2, y_2)$** in the coordinate plane is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

EXAMPLE 3 Finding the Distance Between Two Points

Find the distance d between the points $(1, 5)$ and $(6, 2)$.

SOLUTION

$$
\begin{aligned}
d &= \sqrt{(1 - 6)^2 + (5 - 2)^2} && \text{The distance formula} \\
&= \sqrt{(-5)^2 + 3^2} \\
&= \sqrt{25 + 9} \\
&= \sqrt{34} \approx 5.83 && \text{Using a calculator}
\end{aligned}
$$

Now try Exercise 11.

Midpoint Formulas

When the endpoints of a segment in a number line are known, we take the average of their coordinates to find the midpoint of the segment.

Midpoint Formula (Number Line)

The **midpoint of the line segment with endpoints a and b** is

$$\frac{a + b}{2}.$$

EXAMPLE 4 Finding the Midpoint of a Line Segment

The midpoint of the line segment with endpoints -9 and 3 on a number line is

$$\frac{(-9) + 3}{2} = \frac{-6}{2} = -3.$$

See Figure P.12. *Now try Exercise 23.*

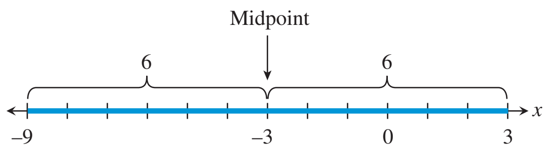


FIGURE P.12 Notice that the distance from the midpoint, -3, to 3 or to -9 is 6. (Example 4)

Just as with number lines, the midpoint of a line segment in the coordinate plane is determined by its endpoints. Each coordinate of the midpoint is the average of the corresponding coordinates of its endpoints.

Midpoint Formula (Coordinate Plane)

The **midpoint of the line segment with endpoints (a, b) and (c, d)** is

$$\left(\frac{a + c}{2}, \frac{b + d}{2}\right).$$

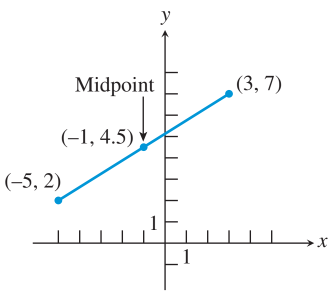


FIGURE P.13 (Example 5.)

EXAMPLE 5 Finding the Midpoint of a Line Segment

The midpoint of the line segment with endpoints $(-5, 2)$ and $(3, 7)$ is

$$(x, y) = \left(\frac{-5 + 3}{2}, \frac{2 + 7}{2}\right) = (-1, 4.5).$$

See Figure P.13. *Now try Exercise 25.*

Equations of Circles

A **circle** is the set of points in a plane at a fixed distance (**radius**) from a fixed point (**center**). Figure P.14 shows the circle with center (h, k) and radius r. If (x, y) is any point on the circle, the distance formula gives

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

Squaring both sides, we obtain the following equation for a circle.

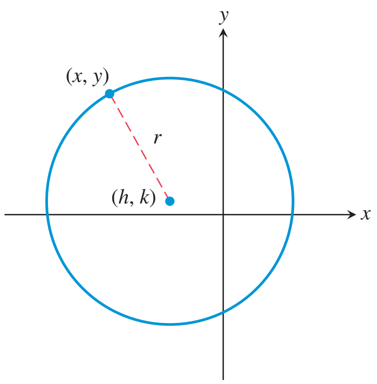


FIGURE P.14 The circle with center (h, k) and radius r.

DEFINITION Standard Form Equation of a Circle

The **standard form equation of a circle** with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

EXAMPLE 6 Finding Standard Form Equations of Circles

Find the standard form equation of the circle.

(a) Center $(-4, 1)$, radius 8 **(b)** Center $(0, 0)$, radius 5

SOLUTION

(a) $(x - h)^2 + (y - k)^2 = r^2$ Standard form equation

$(x - (-4))^2 + (y - 1)^2 = 8^2$ Substitute $h = -4$, $k = 1$, $r = 8$.

$(x + 4)^2 + (y - 1)^2 = 64$

(b) $(x - h)^2 + (y - k)^2 = r^2$ Standard form equation

$(x - 0)^2 + (y - 0)^2 = 5^2$ Substitute $h = 0$, $k = 0$, $r = 5$.

$x^2 + y^2 = 25$

Now try Exercise 41.

Applications

EXAMPLE 7 Using an Inequality to Express Distance

We can state that "the distance between x and -3 is less than 9" using the inequality

$$|x - (-3)| < 9 \quad \text{or} \quad |x + 3| < 9.$$

Now try Exercise 51.

The converse of the Pythagorean Theorem is true. That is, if the sum of squares of the lengths of the two sides of a triangle equals the square of the length of the third side, then the triangle is a right triangle.

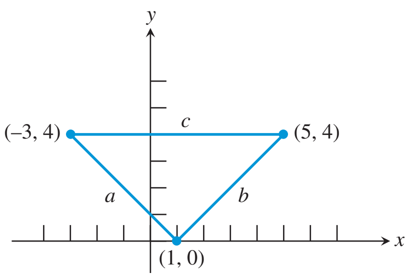


FIGURE P.15 The triangle in Example 8.

EXAMPLE 8 Verifying Right Triangles

Use the converse of the Pythagorean Theorem and the distance formula to show that the points $(-3, 4)$, $(1, 0)$, and $(5, 4)$ determine a right triangle.

SOLUTION The three points are plotted in Figure P.15. We need to show that the lengths of the sides of the triangle satisfy the Pythagorean relationship $a^2 + b^2 = c^2$. Applying the distance formula we find that

$$a = \sqrt{(-3 - 1)^2 + (4 - 0)^2} = \sqrt{32},$$
$$b = \sqrt{(1 - 5)^2 + (0 - 4)^2} = \sqrt{32},$$
$$c = \sqrt{(-3 - 5)^2 + (4 - 4)^2} = \sqrt{64}.$$

The triangle is a right triangle because

$$a^2 + b^2 = (\sqrt{32})^2 + (\sqrt{32})^2 = 32 + 32 = 64 = c^2.$$

Now try Exercise 39.

Properties of geometric figures can sometimes be confirmed using analytic methods such as the midpoint formulas.

EXAMPLE 9 Using the Midpoint Formula

It is a fact from geometry that the diagonals of a parallelogram bisect each other. Prove this with a midpoint formula.

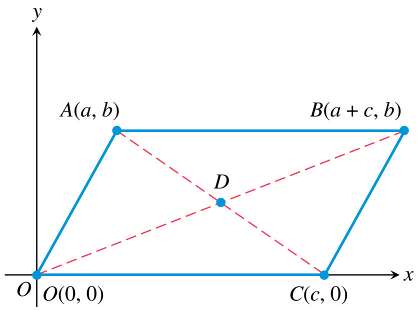


FIGURE P.16 The coordinates of B must be $(a + c, b)$ in order for CB to be parallel to OA. (Example 9)

SOLUTION We can position a parallelogram in the rectangular coordinate plane as shown in Figure P.16. Applying the midpoint formula for the coordinate plane to segments OB and AC, we find that

$$\text{midpoint of segment } OB = \left(\frac{0 + a + c}{2}, \frac{0 + b}{2}\right) = \left(\frac{a + c}{2}, \frac{b}{2}\right),$$

$$\text{midpoint of segment } AC = \left(\frac{a + c}{2}, \frac{b + 0}{2}\right) = \left(\frac{a + c}{2}, \frac{b}{2}\right).$$

The midpoints of segments OA and AC are the same, so the diagonals of the parallelogram $OABC$ meet at their midpoints and thus bisect each other.

Now try Exercise 37.

QUICK REVIEW P.2

In Exercises 1 and 2, plot the two numbers on a number line. Then find the distance between them.

1. $\sqrt{7}, \sqrt{2}$ **2.** $-\frac{5}{3}, -\frac{9}{5}$

In Exercises 3 and 4, plot the real numbers on a number line.

3. $-3, 4, 2.5, 0, -1.5$ **4.** $-\frac{5}{2}, -\frac{1}{2}, \frac{2}{3}, 0, -1$

In Exercises 5 and 6, plot the points.

5. $A(3, 5), B(-2, 4), C(3, 0), D(0, -3)$

6. $A(-3, -5), B(2, -4), C(0, 5), D(-4, 0)$

In Exercises 7–10, use a calculator to evaluate the expression. Round your answer to two decimal places.

7. $\frac{-17 + 28}{2}$ **8.** $\sqrt{13^2 + 17^2}$

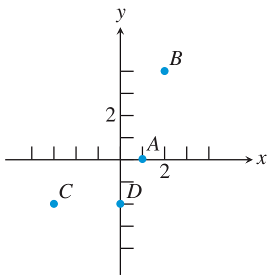
9. $\sqrt{6^2 + 8^2}$ **10.** $\sqrt{(17 - 3)^2 + (-4 - 8)^2}$

SECTION P.2 EXERCISES

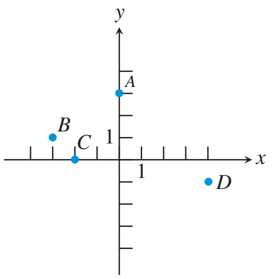
Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, estimate the coordinates of the points.

1.



2.



In Exercises 3 and 4, name the quadrants containing the points.

3. **(a)** $(2, 4)$ **(b)** $(0, 3)$ **(c)** $(-2, 3)$ **(d)** $(-1, -4)$

4. **(a)** $\left(\frac{1}{2}, \frac{3}{2}\right)$ **(b)** $(-2, 0)$ **(c)** $(-1, -2)$ **(d)** $\left(-\frac{3}{2}, -\frac{7}{3}\right)$

In Exercises 5–8, evaluate the expression.

5. $3 + |-3|$ **6.** $2 - |-2|$

7. $|(-2)3|$ **8.** $\frac{-2}{|-2|}$

In Exercises 9 and 10, rewrite the expression without using absolute value symbols.

9. $|\pi - 4|$ **10.** $|\sqrt{5} - 5/2|$

In Exercises 11–18, find the distance between the points.

11. $-9.3, 10.6$ **12.** $-5, -17$

13. $(-3, -1), (5, -1)$ **14.** $(-4, -3), (1, 1)$

15. $(0, 0), (3, 4)$ **16.** $(-1, 2), (2, -3)$

17. $(-2, 0), (5, 0)$ **18.** $(0, -8), (0, -1)$

In Exercises 19–22, find the area and perimeter of the figure determined by the points.

19. $(-5, 3), (0, -1), (4, 4)$

20. $(-2, -2), (-2, 2), (2, 2), (2, -2)$

21. $(-3, -1), (-1, 3), (7, 3), (5, -1)$

22. $(-2, 1), (-2, 6), (4, 6), (4, 1)$

In Exercises 23–28, find the midpoint of the line segment with the given endpoints.

23. $-9.3, 10.6$ **24.** $-5, -17$

25. $(-1, 3), (5, 9)$

26. $(3, \sqrt{2}), (6, 2)$

27. $(-7/3, 3/4), (5/3, -9/4)$

28. $(5, -2), (-1, -4)$

In Exercises 29–34, draw a scatter plot of the data given in the table.

29. U.S. Motor Vehicle Production The total number of motor vehicles in thousands (y) produced by the United States each year from 2001 to 2007 is given in the table. *(Source: Automotive News Data Center and R. L. Polk Marketing Systems as reported in The World Almanac and Book of Facts 2009.)*

x	2001	2002	2003	2004	2005	2006	2007
y	11,518	12,328	12,145	12,021	12,018	11,351	10,611

30. World Motor Vehicle Production The total number of motor vehicles in thousands (y) produced in the world each year from 2001 to 2007 is given in the table. *(Source: American Automobile Manufacturers Association as reported in The World Almanac and Book of Facts 2009.)*

x	2001	2002	2003	2004	2005	2006	2007
y	57,705	59,587	61,562	65,654	67,892	70,992	74,647

31. U.S. Imports from Mexico The total in billions of dollars of U.S. imports from Mexico from 2000 to 2007 is given in Table P.3.

Table P.3 U.S. Imports from Mexico

Year	U.S. Imports (billions of dollars)
2000	135.0
2001	131.3
2002	134.6
2003	138.1
2004	155.9
2005	170.1
2006	188.2
2007	210.7

Source: U.S. Census Bureau, The World Almanac and Book of Facts 2009.

32. U.S. Agricultural Exports The total in billions of dollars of U.S. agricultural exports from 2000 to 2007 is given in Table P.4.

Table P.4 U.S. Agricultural Exports

Year	U.S. Agricultural Exports (billions of dollars)
2000	51.2
2001	53.7
2002	53.1
2003	56.0
2004	62.4
2005	62.5
2006	68.7
2007	89.2

Source: U.S. Department of Agriculture, The World Almanac and Book of Facts 2009.

33. U.S. Agricultural Trade Surplus The total in billions of dollars of U.S. agricultural trade surplus from 2000 to 2007 is given in Table P.5.

Table P.5 U.S. Agricultural Trade Surplus

Year	U.S. Agricultural Trade Surplus (billions of dollars)
2000	12.2
2001	14.3
2002	11.2
2003	10.3
2004	9.7
2005	4.8
2006	4.7
2007	12.1

Source: U.S. Department of Agriculture, The World Almanac and Book of Facts 2009.

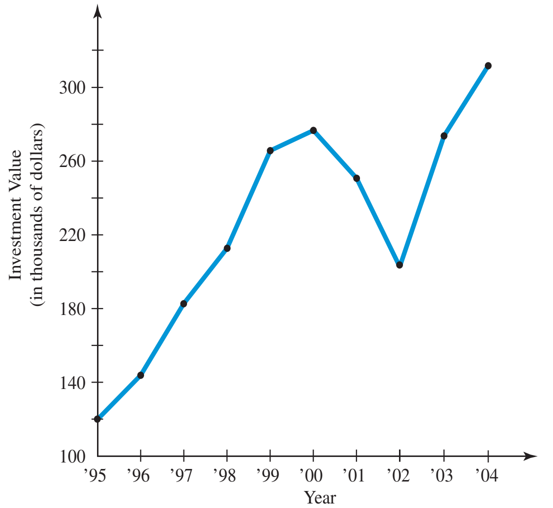
34. U.S. Exports to Canada The total in billions of dollars of U.S. exports to Canada from 2000 to 2007 is given in Table P.6.

Table P.6 U.S. Exports to Canada

Year	U.S. Exports (billions of dollars)
2000	178.9
2001	163.4
2002	160.9
2003	169.9
2004	189.9
2005	211.9
2006	230.6
2007	248.9

Source: U.S. Census Bureau, The World Almanac and Book of Facts 2009.

In Exercises 35 and 36, use the graph of the investment value of a \$10,000 investment made in 1978 in Fundamental Investors™ of the American Funds™. The value as of January is shown for a few recent years in the graph below. (*Source: Annual report of Fundamental Investors for the year ending December 31, 2004.*)



35. Reading from Graphs Use the graph to estimate the value of the investment as of

(a) January 1997 and **(b)** January 2000.

36. Percent Increase Estimate the percent increase in the value of the \$10,000 investment from

(a) January 1996 to January 1997.

(b) January 2000 to January 2001.

(c) January 1995 to January 2004.

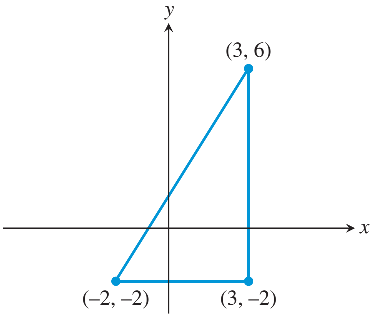
37. Prove that the figure determined by the points is an isosceles triangle: $(1, 3)$, $(4, 7)$, $(8, 4)$

38. Group Activity Prove that the diagonals of the figure determined by the points bisect each other.

(a) Square $(-7, -1)$, $(-2, 4)$, $(3, -1)$, $(-2, -6)$

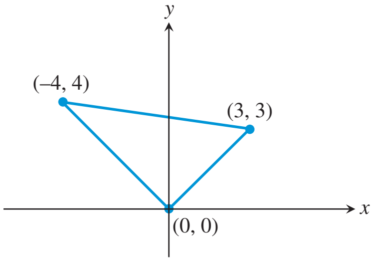
(b) Parallelogram $(-2, -3)$, $(0, 1)$, $(6, 7)$, $(4, 3)$

39. (a) Find the lengths of the sides of the triangle in the figure.



(b) Writing to Learn Show that the triangle is a right triangle.

40. (a) Find the lengths of the sides of the triangle in the figure.



(b) Writing to Learn Show that the triangle is a right triangle.

In Exercises 41–44, find the standard form equation for the circle.

41. Center $(1, 2)$, radius 5

42. Center $(-3, 2)$, radius 1

43. Center $(-1, -4)$, radius 3

44. Center $(0, 0)$, radius $\sqrt{3}$

In Exercises 45–48, find the center and radius of the circle.

45. $(x - 3)^2 + (y - 1)^2 = 36$

46. $(x + 4)^2 + (y - 2)^2 = 121$

47. $x^2 + y^2 = 5$

48. $(x - 2)^2 + (y + 6)^2 = 25$

In Exercises 49–52, write the statement using absolute value notation.

49. The distance between x and 4 is 3.

50. The distance between y and -2 is greater than or equal to 4.

51. The distance between x and c is less than d units.

52. y is more than d units from c.

53. Determining a Line Segment with Given Midpoint Let $(4, 4)$ be the midpoint of the line segment determined by the points $(1, 2)$ and (a, b). Determine a and b.

54. Writing to Learn Isosceles but Not Equilateral Triangle Prove that the triangle determined by the points $(3, 0)$, $(-1, 2)$, and $(5, 4)$ is isosceles but not equilateral.

55. Writing to Learn Equidistant Point from Vertices of a Right Triangle Prove that the midpoint of the hypotenuse of the right triangle with vertices $(0, 0)$, $(5, 0)$, and $(0, 7)$ is equidistant from the three vertices.

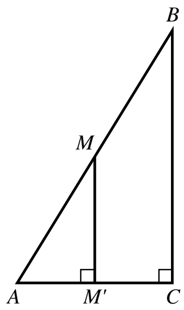
56. Writing to Learn Describe the set of real numbers that satisfy $|x - 2| < 3$.

57. Writing to Learn Describe the set of real numbers that satisfy $|x + 3| \geq 5$.

Standardized Test Questions

58. True or False If a is a real number, then $|a| \geq 0$. Justify your answer.

59. True or False Consider the right triangle ABC shown at the right. If M is the midpoint of the segment AB, then M' is the midpoint of the segment AC. Justify your answer.



In Exercises 60–63, solve these problems without using a calculator.

60. Multiple Choice Which of the following is equal to $|1 - \sqrt{3}|$?

(A) $1 - \sqrt{3}$ **(B)** $\sqrt{3} - 1$

(C) $(1 - \sqrt{3})^2$ **(D)** $\sqrt{2}$

(E) $\sqrt{1/3}$

61. Multiple Choice Which of the following is the midpoint of the line segment with endpoints -3 and 2?

(A) $5/2$ **(B)** 1

(C) $-1/2$ **(D)** -1

(E) $-5/2$

62. Multiple Choice Which of the following is the center of the circle $(x - 3)^2 + (y + 4)^2 = 2$?

(A) $(3, -4)$
(B) $(-3, 4)$
(C) $(4, -3)$
(D) $(-4, 3)$
(E) $(3/2, -2)$

63. Multiple Choice Which of the following points is in the third quadrant?

(A) $(0, -3)$
(B) $(-1, 0)$
(C) $(2, -1)$
(D) $(-1, 2)$
(E) $(-2, -3)$

Explorations

64. Dividing a Line Segment into Thirds

(a) Find the coordinates of the points one-third and two-thirds of the way from $a = 2$ to $b = 8$ on a number line.

(b) Repeat (a) for $a = -3$ and $b = 7$.

(c) Find the coordinates of the points one-third and two-thirds of the way from a to b on a number line.

(d) Find the coordinates of the points one-third and two-thirds of the way from the point $(1, 2)$ to the point $(7, 11)$ in the coordinate plane.

(e) Find the coordinates of the points one-third and two-thirds of the way from the point (a, b) to the point (c, d) in the coordinate plane.

Extending the Ideas

65. Writing to Learn Equidistant Point from Vertices of a Right Triangle Prove that the midpoint of the hypotenuse of any right triangle is equidistant from the three vertices.

66. Comparing Areas Consider the four points $A(0, 0)$, $B(0, a)$, $C(a, a)$, and $D(a, 0)$. Let P be the midpoint of the line segment CD and Q the point one-fourth of the way from A to D on segment AD.

(a) Find the area of triangle BPQ.

(b) Compare the area of triangle BPQ with the area of square $ABCD$.

In Exercises 67–69, let $P(a, b)$ be a point in the first quadrant.

67. Find the coordinates of the point Q in the fourth quadrant so that PQ is perpendicular to the x-axis.

68. Find the coordinates of the point Q in the second quadrant so that PQ is perpendicular to the y-axis.

69. Find the coordinates of the point Q in the third quadrant so that the origin is the midpoint of the segment PQ.

70. Writing to Learn Prove that the distance formula for the number line is a special case of the distance formula for the Cartesian plane.