9.5 Series

What you'll learn about

- Summation Notation
- Sums of Arithmetic and Geometric Sequences
- Infinite Series
- Convergence of Geometric Series

. . . and why

Infinite series are at the heart of integral calculus.

Summation Notation

We want to look at the formulas for summing the terms of arithmetic and geometric sequences, but first we need a notation for writing the sum of an indefinite number of terms. The capital Greek letter sigma (Σ) provides our shorthand notation for a "summation."

DEFINITION Summation Notation

In **summation notation**, the sum of the terms of the sequence $\{a_1, a_2, \ldots, a_n\}$ is denoted

$$\sum_{k=1}^{n} a_k$$

which is read "the sum of a_k from $k = 1$ to n."

The variable k is called the **index of summation**.

Summations on a Calculator

If you think of summations as summing sequence values, it is not hard to translate sigma notation into calculator syntax. Here, in calculator syntax, are the first three summations in Exploration 1. (Don't try these on your calculator until you have first figured out the answers with pencil and paper.)

1. sum (seq (3K, K, 1, 5))

2. sum (seq (K^2, K, 5, 8))

3. sum (seq (cos (Nπ), N, 0, 12))

EXPLORATION 1 Summing with Sigma

Sigma notation is actually even more versatile than the definition above suggests. See if you can determine the number represented by each of the following expressions.

1. $\sum_{k=1}^{5} 3k$ **2.** $\sum_{k=5}^{8} k^2$ **3.** $\sum_{n=0}^{12} \cos(n\pi)$ **4.** $\sum_{n=1}^{\infty} \sin(n\pi)$ **5.** $\sum_{k=1}^{\infty} \frac{3}{10^k}$

(If you're having trouble with number 5, here's a hint: Write the sum as a decimal!)

Although you probably computed them correctly, there is more going on in number 4 and number 5 in the above exploration than first meets the eye. We will have more to say about these "infinite" summations toward the end of this section.

Sums of Arithmetic and Geometric Sequences

One of the most famous legends in the lore of mathematics concerns the German mathematician Karl Friedrich Gauss (1777–1855), whose mathematical talent was apparent at a very early age. One version of the story has Gauss, at age ten, being in a class that was challenged by the teacher to add up all the numbers from 1 to 100. While his classmates were still writing down the problem, Gauss walked to the front of the room to present his slate to the teacher. The teacher, certain that Gauss could only be guessing, refused to look at his answer. Gauss simply placed it face down on the teacher's desk, declared "There it is," and returned to his seat. Later, after all the slates had been collected, the teacher looked at Gauss's work, which consisted of a single number: the correct answer. No other student (the legend goes) got it right.

The important feature of this legend for mathematicians is *how* the young Gauss got the answer so quickly. We'll let you reproduce his technique in Exploration 2.

EXPLORATION 2 Gauss's Insight

Your challenge is to find the sum of the natural numbers from 1 to 100 without a calculator.

1. On a wide piece of paper, write the sum

"$1 + 2 + 3 + \cdots + 98 + 99 + 100$."

2. Underneath this sum, write the sum

"$100 + 99 + 98 + \cdots + 3 + 2 + 1$."

3. Add the numbers two-by-two in *vertical* columns and notice that you get the same identical sum 100 times. What is it?
4. What is the sum of the 100 identical numbers referred to in part 3?
5. Explain why half the answer in part 4 is the answer to the challenge. Can you find it without a calculator?

If this story is true, then the youthful Gauss had discovered a fact that his elders knew about arithmetic sequences. If you write a finite arithmetic sequence forward on one line and backward on the line below it, then all the pairs stacked vertically sum to the same number. Multiplying this number by the number of terms n and dividing by 2 gives us a shortcut to the sum of the n terms. We state this result as a theorem.

THEOREM Sum of a Finite Arithmetic Sequence

Let $\{a_1, a_2, \ldots, a_n\}$ be a finite arithmetic sequence with common difference d. Then the sum of the terms of the sequence is

$$\begin{aligned}\sum_{k=1}^{n} a_k &= a_1 + a_2 + \cdots + a_n \\ &= n\left(\frac{a_1 + a_n}{2}\right) \\ &= \frac{n}{2}(2a_1 + (n-1)d)\end{aligned}$$

Proof

We can construct the sequence forward by starting with a_1 and *adding* d each time, or we can construct the sequence backward by starting at a_n and *subtracting* d each time. We thus get two expressions for the sum we are looking for:

$$\sum_{k=1}^{n} a_k = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-1)d)$$

$$\sum_{k=1}^{n} a_k = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-1)d)$$

Summing vertically, we get

$$\begin{aligned}2\sum_{k=1}^{n} a_k &= (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \\ 2\sum_{k=1}^{n} a_k &= n(a_1 + a_n) \\ \sum_{k=1}^{n} a_k &= n\left(\frac{a_1 + a_n}{2}\right)\end{aligned}$$

If we substitute $a_1 + (n - 1)d$ for a_n, we get the alternate formula

$$\sum_{k=1}^{n} a_k = \frac{n}{2}(2a_1 + (n - 1)d).$$

EXAMPLE 1 Summing the Terms of an Arithmetic Sequence

A corner section of a stadium has 8 seats along the front row. Each successive row has two more seats than the row preceding it. If the top row has 24 seats, how many seats are in the entire section?

SOLUTION The numbers of seats in the rows form an arithmetic sequence with

$$a_1 = 8, \quad a_n = 24, \quad \text{and} \quad d = 2.$$

Solving $a_n = a_1 + (n - 1)d$, we find that

$$\begin{aligned} 24 &= 8 + (n - 1)(2) \\ 16 &= (n - 1)(2) \\ 8 &= n - 1 \\ n &= 9 \end{aligned}$$

Applying the Sum of a Finite Arithmetic Sequence Theorem, the total number of seats in the section is $9(8 + 24)/2 = 144$.

We can support this answer numerically by computing the sum on a calculator:

$$\text{sum}(\text{seq}(8 + (\text{N} - 1)2, \text{N}, 1, 9) = 144$$

Now try Exercise 7.

As you might expect, there is also a convenient formula for summing the terms of a finite geometric sequence.

THEOREM Sum of a Finite Geometric Sequence

Let $\{a_1, a_2, a_3, \ldots, a_n\}$ be a finite geometric sequence with common ratio $r \neq 1$.

Then the sum of the terms of the sequence is

$$\begin{aligned} \sum_{k=1}^{n} a_k &= a_1 + a_2 + \cdots + a_n \\ &= \frac{a_1(1 - r^n)}{1 - r}. \end{aligned}$$

Proof

Because the sequence is geometric, we have

$$\sum_{k=1}^{n} a_k = a_1 + a_1 \cdot r + a_1 \cdot r^2 + \cdots + a_1 \cdot r^{n-1}.$$

Therefore,

$$r \cdot \sum_{k=1}^{n} a_k = a_1 \cdot r + a_1 \cdot r^2 + \cdots + a_1 \cdot r^{n-1} + a_1 \cdot r^n.$$

If we now *subtract* the lower summation from the one above it, we have (after eliminating a lot of zeros):

$$\left(\sum_{k=1}^{n} a_k\right) - r\cdot\left(\sum_{k=1}^{n} a_k\right) = a_1 - a_1\cdot r^n$$

$$\left(\sum_{k=1}^{n} a_k\right)(1 - r) = a_1(1 - r^n)$$

$$\sum_{k=1}^{n} a_k = \frac{a_1(1 - r^n)}{1 - r}$$

EXAMPLE 2 Summing the Terms of a Geometric Sequence

Find the sum of the geometric sequence $4, -4/3, 4/9, -4/27, \ldots, 4(-1/3)^{10}$.

SOLUTION We can see that $a_1 = 4$ and $r = -1/3$. The nth term is $4(-1/3)^{10}$, which means that $n = 11$. (Remember that the exponent on the nth term is $n - 1$, not n.) Applying the Sum of a Finite Geometric Sequence Theorem, we find that

$$\sum_{n=1}^{11} 4\left(-\frac{1}{3}\right)^{n-1} = \frac{4(1 - (-1/3)^{11})}{1 - (-1/3)} \approx 3.000016935.$$

We can support this answer by having the calculator do the actual summing: $\text{sum}(\text{seq}(4(-1/3)\text{^}(N - 1), N, 1, 11) = 3.000016935.$ ***Now try Exercise 13.***

As one practical application of the Sum of a Finite Geometric Sequence Theorem, we will tie up a loose end from Section 3.6, wherein you learned that the future value FV of an ordinary annuity consisting of n equal periodic payments of R dollars at an interest rate i per compounding period (payment interval) is

$$FV = R\frac{(1 + i)^n - 1}{i}.$$

We can now consider the mathematics behind this formula. The n payments remain in the account for different lengths of time and so earn different amounts of interest. The total value of the annuity after n payment periods (see Example 8 in Section 3.6) is

$$FV = R + R(1 + i) + R(1 + i)^2 + \cdots + R(1 + i)^{n-1}.$$

The terms of this sum form a geometric sequence with first term R and common ratio $(1 + i)$. Applying the Sum of a Finite Geometric Sequence Theorem, the sum of the n terms is

$$\begin{aligned} FV &= \frac{R(1 - (1 + i)^n)}{1 - (1 + i)} \\ &= R\frac{1 - (1 + i)^n}{-i} \\ &= R\frac{(1 + i)^n - 1}{i} \end{aligned}$$

Infinite Series

If you change the "11" in the calculator sum in Example 2 to higher and higher numbers, you will find that the sum approaches a value of 3. This is no coincidence. In the language of limits,

$$\begin{aligned} \lim_{x\to\infty} \sum_{k=1}^{n} 4\left(-\frac{1}{3}\right)^{k-1} &= \lim_{x\to\infty} \frac{4(1 - (-1/3)^n)}{1 - (-1/3)} \\ &= \frac{4(1 - 0)}{4/3} \\ &= 3 \end{aligned}$$

This gives us the opportunity to extend the usual meaning of the word "sum," which always applies to a *finite* number of terms being added together. By using limits, we can make sense of expressions in which an *infinite* number of terms are added together. Such expressions are called **infinite series**.

DEFINITION Infinite Series

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots.$$

The first thing to understand about an infinite series is that it is not a true sum. There are properties of real number addition that allow us to extend the definition of $a + b$ to sums like $a + b + c + d + e + f$, but not to "infinite sums." For example, we can add any finite number of 2's together and get a real number, but if we add an *infinite* number of 2's together we do not get a real number at all. Sums do not behave that way.

What makes series so interesting is that sometimes (as in Example 2) the sequence of **partial sums**, all of which are true sums, approaches a finite limit S:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = S$$

In this case we say that the series **converges** to S, and it makes sense to define S as the **sum of the infinite series**. In sigma notation,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = S.$$

If the limit of partial sums does not exist, then the series **diverges** and has no sum.

EXAMPLE 3 Looking at Limits of Partial Sums

For each of the following series, find the first five terms in the sequence of partial sums. Which of the series appear to converge?

(a) $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

(b) $10 + 20 + 30 + 40 + \cdots$

(c) $1 - 1 + 1 - 1 + \cdots$

SOLUTION

(a) The first five partial sums are $\{0.1, 0.11, 0.111, 0.1111, 0.11111\}$. These appear to be approaching a limit of $0.\overline{1} = 1/9$, which would suggest that the series converges to a sum of $1/9$.

(b) The first five partial sums are $\{10, 30, 60, 100, 150\}$. These numbers increase without bound and do not approach a limit. The series diverges and has no sum.

(c) The first five partial sums are $\{1, 0, 1, 0, 1\}$. These numbers oscillate and do not approach a limit. The series diverges and has no sum.

Now try Exercise 23.

You might have been tempted to "pair off" the terms in Example 3c to get an infinite summation of 0's (and hence a sum of 0), but you would be applying a rule (namely the *associative property of addition*) that works on *finite* sums but not, in general, on infinite series. The sequence of partial sums does not have a limit, so any manipulation of the series in Example 3c that appears to result in a sum is actually meaningless.

Convergence of Geometric Series

Determining the convergence or divergence of infinite series is an important part of a calculus course, in which series are used to represent functions. Most of the convergence tests are well beyond the scope of this course, but we are in a position to settle the issue completely for geometric series.

THEOREM Sum of an Infinite Geometric Series

The geometric series $\sum_{k=1}^{\infty} a \cdot r^{k-1}$ converges if and only if $|r| < 1$. If it does converge, the sum is $a/(1-r)$.

Proof

If $r = 1$, the series is $a + a + a + \cdots$, which is unbounded and hence diverges. If $r = -1$, the series is $a - a + a - a + \cdots$, which diverges. (See Example 3c.) If $r \neq 1$, then by the Sum of a Finite Geometric Sequence Theorem, the nth partial sum of the series is $\sum_{k=1}^{n} a \cdot r^{k-1} = a(1-r^n)/(1-r)$. The limit of the partial sums is $\lim_{n\to\infty}[a(1-r^n)/(1-r)]$, which converges if and only if $\lim_{n\to\infty} r^n$ is a finite number. But $\lim_{n\to\infty} r^n$ is 0 when $|r| < 1$ and unbounded when $|r| > 1$. Therefore, the sequence of partial sums converges if and only if $|r| < 1$, in which case the sum of the series is

$$\lim_{n\to\infty}[a(1-r^n)/(1-r)] = a(1-0)/(1-r) = a/(1-r).$$

EXAMPLE 4 Summing Infinite Geometric Series

Determine whether the series converges. If it converges, give the sum.

(a) $\displaystyle\sum_{k=1}^{\infty} 3(0.75)^{k-1}$ **(b)** $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\pi}{2}\right)^n$ **(d)** $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots$

SOLUTION

(a) Since $|r| = |0.75| < 1$, the series converges. The first term is $3(0.75)^0 = 3$, so the sum is $a/(1-r) = 3/(1-0.75) = 12$.

(b) Since $|r| = |-4/5| < 1$, the series converges. The first term is $(-4/5)^0 = 1$, so the sum is $a/(1-r) = 1/(1-(-4/5)) = 5/9$.

(c) Since $|r| = |\pi/2| > 1$, the series diverges.

(d) Since $|r| = |1/2| < 1$, the series converges. The first term is 1, and so the sum is $a/(1-r) = 1/(1-1/2) = 2$.

Now try Exercise 25.

EXAMPLE 5 Converting a Repeating Decimal to Fraction Form

Express $0.\overline{234} = 0.234234234\ldots$ in fraction form.

SOLUTION We can write this number as a sum: $0.234 + 0.000234 + 0.000000234 + \cdots$.

This is a convergent infinite geometric series in which $a = 0.234$ and $r = 0.001$. The sum is

$$\frac{a}{1-r} = \frac{0.234}{1-0.001} = \frac{0.234}{0.999} = \frac{234}{999} = \frac{26}{111}.$$

Now try Exercise 31.

QUICK REVIEW 9.5 *(For help, see Section 9.4.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, $\{a_n\}$ is arithmetic. Use the given information to find a_{10}.

1. $a_1 = 4; d = 2$

2. $a_1 = 3; a_2 = 1$

3. $a_3 = 6; a_8 = 21$

4. $a_5 = 3; a_{n+1} = a_n + 5$ for $n \geq 1$

In Exercises 5–8, $\{a_n\}$ is geometric. Use the given information to find a_{10}.

5. $a_1 = 1; a_2 = 2$

6. $a_4 = 1; a_6 = 2$

7. $a_7 = 5; r = -2$

8. $a_8 = 10; a_{12} = 40$

9. Find the sum of the first 5 terms of the sequence $\{n^2\}$.

10. Find the sum of the first 5 terms of the sequence $\{2n - 1\}$.

SECTION 9.5 EXERCISES

In Exercises 1–6, write each sum using summation notation, assuming the suggested pattern continues.

1. $-7 - 1 + 5 + 11 + \cdots + 53$

2. $2 + 5 + 8 + 11 + \cdots + 29$

3. $1 + 4 + 9 + \cdots + (n + 1)^2$

4. $1 + 8 + 27 + \cdots + (n + 1)^3$

5. $6 - 12 + 24 - 48 + \cdots$

6. $5 - 15 + 45 - 135 + \cdots$

In Exercises 7–12, find the sum of the arithmetic sequence.

7. $-7, -3, 1, 5, 9, 13$

8. $-8, -1, 6, 13, 20, 27$

9. $1, 2, 3, 4, \ldots, 80$

10. $2, 4, 6, 8, \ldots, 70$

11. $117, 110, 103, \ldots, 33$

12. $111, 108, 105, \ldots, 27$

In Exercises 13–16, find the sum of the geometric sequence.

13. $3, 6, 12, \ldots, 12{,}288$

14. $5, 15, 45, \ldots, 98{,}415$

15. $42, 7, \frac{7}{6}, \ldots, 42\left(\frac{1}{6}\right)^8$

16. $42, -7, \frac{7}{6}, \ldots, 42\left(-\frac{1}{6}\right)^9$

In Exercises 17–22, find the sum of the first n terms of the sequence. The sequence is either arithmetic or geometric.

17. $2, 5, 8, \ldots; n = 10$

18. $14, 8, 2, \ldots; n = 9$

19. $4, -2, 1, -\frac{1}{2}, \ldots; n = 12$

20. $6, -3, \frac{3}{2}, -\frac{3}{4}, \ldots; n = 11$

21. $-1, 11, -121, \ldots; n = 9$

22. $-2, 24, -288, \ldots; n = 8$

23. Find the first six partial sums of the following infinite series. If the sums have a finite limit, write "convergent." If not, write "divergent."

(a) $0.3 + 0.03 + 0.003 + 0.0003 + \cdots$

(b) $1 - 2 + 3 - 4 + 5 - 6 + \cdots$

24. Find the first six partial sums of the following infinite series. If the sums have a finite limit, write "convergent." If not, write "divergent."

(a) $-2 + 2 - 2 + 2 - 2 + \cdots$

(b) $1 - 0.7 - 0.07 - 0.007 - 0.0007 - \cdots$

In Exercises 25–30, determine whether the infinite geometric series converges. If it does, find its sum.

25. $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$

26. $4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots$

27. $\frac{1}{64} + \frac{1}{32} + \frac{1}{16} + \frac{1}{8} + \cdots$

28. $\frac{1}{48} + \frac{1}{16} + \frac{3}{16} + \frac{9}{16} + \cdots$

29. $\sum_{j=1}^{\infty} 3\left(\frac{1}{4}\right)^j$

30. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$

In Exercises 31–34, express the rational number as a fraction of integers.

31. $7.14141414\ldots$

32. $5.93939393\ldots$

33. $-17.268268268\ldots$

34. $-12.876876876\ldots$

35. Savings Account The table below shows the December balance in a fixed-rate compound savings account each year from 1996 to 2000.

Year	1996	1997	1998	1999	2000
Balance	\$20,000	\$22,000	\$24,200	\$26,620	\$29,282

(a) The balances form a geometric sequence. What is r?

(b) Write a formula for the balance in the account n years after December 1996.

(c) Find the sum of the December balances from 1996 to 2006, inclusive.

36. Savings Account The table below shows the December balance in a simple interest savings account each year from 1996 to 2000.

Year	1996	1997	1998	1999	2000
Balance	\$18,000	\$20,016	\$22,032	\$24,048	\$26,064

(a) The balances form an arithmetic sequence. What is d?

(b) Write a formula for the balance in the account n years after December 1996.

(c) Find the sum of the December balances from 1996 to 2006, inclusive.

37. Annuity Mr. O'Hara deposits \$120 at the end of each month into an account that pays 7% interest compounded monthly. After 10 years, the balance in the account, in dollars, is

$$120\left(1+\frac{0.07}{12}\right)^0 + 120\left(1+\frac{0.07}{12}\right)^1 + \cdots + 120\left(1+\frac{0.07}{12}\right)^{119}.$$

(a) This is a geometric series. What is the first term? What is r?

(b) Use the formula for the sum of a finite geometric sequence to find the balance.

38. Annuity Ms. Argentieri deposits \$100 at the end of each month into an account that pays 8% interest compounded monthly. After 10 years, the balance in the account, in dollars, is

$$100\left(1+\frac{0.08}{12}\right)^0 + 100\left(1+\frac{0.08}{12}\right)^1 + \cdots + 100\left(1+\frac{0.08}{12}\right)^{119}.$$

(a) This is a geometric series. What is the first term? What is r?

(b) Use the formula for the sum of a finite geometric sequence to find the balance.

39. Group Activity Follow the Bouncing Ball When "superballs" sprang upon the scene in the 1960s, kids across the United States were amazed that these hard rubber balls could bounce to 90% of the height from which they were dropped. If a superball is dropped from a height of 2 m, how far does it travel until it stops bouncing? [*Hint:* The ball goes down to the first bounce, then up *and* down thereafter.]

40. Writing to Learn The Trouble with Flubber In the 1961 movie classic *The Absent Minded Professor*, Prof. Ned Brainard discovers flubber (flying rubber). If a "super duper ball" made of flubber is dropped, it rebounds to an ever greater height with each bounce. How far does it travel if allowed to keep bouncing?

Standardized Test Questions

41. True or False If all terms of a series are positive, the series sums to a positive number. Justify your answer.

42. True or False If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges. Justify your answer.

You should solve Exercises 43–46 without the use of a calculator.

43. Multiple Choice The series $3^{-1} + 3^{-2} + 3^{-3} + \cdots + 3^{-n} + \cdots$

(A) Converges to 1/2. **(B)** Converges to 1/3.
(C) Converges to 2/3. **(D)** Converges to 3/2. **(E)** Diverges.

44. Multiple Choice If $\sum_{n=1}^{\infty} x^n = 4$, then $x =$

(A) 0.2. **(B)** 0.25. **(C)** 0.4. **(D)** 0.8. **(E)** 4.0.

45. Multiple Choice The sum of an infinite geometric series with first term 3 and second term 0.75 is

(A) 3.75. **(B)** 2.4. **(C)** 4. **(D)** 5. **(E)** 12.

46. Multiple Choice $\sum_{n=0}^{\infty} 4\left(-\frac{5}{3}\right)^n =$

(A) -6 **(B)** $-\frac{5}{2}$ **(C)** $\frac{3}{2}$ **(D)** 10 **(E)** Divergent

Explorations

47. Population Density The *National Geographic Picture Atlas of Our Fifty States* (2001) groups the states into 10 regions. The two largest groupings are the Heartland (Table 9.1) and the Southeast (Table 9.2). Population and area data for the two regions are given in the tables. The populations are official 2000 U.S. Census figures.

(a) What is the total population of each region?

(b) What is the total area of each region?

(c) What is the population density (in persons per square mile) of each region?

(d) Writing to Learn For the two regions, compute the population density of each state. What is the average of the seven state population densities for each region? Explain why these answers differ from those found in part (c).

Table 9.1 The Heartland

State	Population	Area (mi^2)
Iowa	2,926,324	56,275
Kansas	2,688,418	82,277
Minnesota	4,919,479	84,402
Missouri	5,595,211	69,697
Nebraska	1,711,283	77,355
North Dakota	642,200	70,703
South Dakota	754,844	77,116

Table 9.2 The Southeast

State	Population	Area (mi^2)
Alabama	4,447,100	51,705
Arkansas	2,673,400	53,187
Florida	15,982,378	58,644
Georgia	8,186,453	58,910
Louisiana	4,468,976	47,751
Mississippi	2,844,658	47,689
S. Carolina	4,012,012	31,113

48. Finding a Pattern Write the finite series $-1 + 2 + 7 + 14 + 23 + \cdots + 62$ in summation notation.

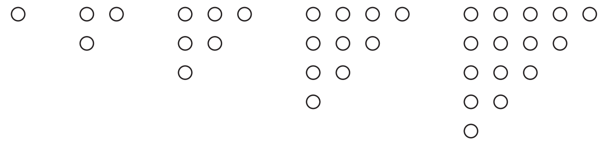
Extending the Ideas

49. Fibonacci Sequence and Series Complete the following table, where F_n is the nth term of the Fibonacci sequence and S_n is the nth partial sum of the Fibonacci series. Make a conjecture based on the numerical evidence in the table.

$$S_n = \sum_{k=1}^{n} F_k$$

n	F_n	S_n	$F_{n+2} - 1$
1	1		
2	1		
3	2		
4			
5			
6			
7			
8			
9			

50. Triangular Numbers Revisited Exercise 41 in Section 9.2 introduced **triangular numbers** as numbers that count objects arranged in triangular arrays:



1 3 6 10 15

In that exercise, you gave a geometric argument that the nth triangular number was $n(n + 1)/2$. Prove that formula algebraically using the Sum of a Finite Arithmetic Sequence Theorem.

51. Square Numbers and Triangular Numbers Prove that the sum of two consecutive triangular numbers is a square number; that is, prove

$$T_{n-1} + T_n = n^2$$

for all positive integers $n \geq 2$. Use both a geometric and an algebraic approach.

52. Harmonic Series Graph the sequence of partial sums of the *harmonic series:*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

Overlay on it the graph of $f(x) = \ln x$. The resulting picture should support the claim that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \geq \ln n,$$

for all positive integers n. Make a table of values to further support this claim. Explain why the claim implies that the harmonic series must diverge.