9.3 Probability

What you'll learn about

- Sample Spaces and Probability Functions
- Determining Probabilities
- Venn Diagrams and Tree Diagrams
- Conditional Probability
- Binomial Distributions

… and why

Everyone should know how mathematical the "laws of chance" really are.

Sample Spaces and Probability Functions

Most people have an intuitive sense of probability. Unfortunately, this sense is not often based on a foundation of mathematical principles, so people become victims of scams, misleading statistics, and false advertising. In this lesson, we want to build on your intuitive sense of probability and give it a mathematical foundation.

EXAMPLE 1 Testing Your Intuition About Probability

Find the probability of each of the following events.

(a) Tossing a head on one toss of a fair coin

(b) Tossing two heads in a row on two tosses of a fair coin

(c) Drawing a queen from a standard deck of 52 cards

(d) Rolling a sum of 4 on a single roll of two fair dice

(e) Guessing all 6 numbers in a state lottery that requires you to pick 6 numbers between 1 and 46, inclusive

SOLUTION

(a) There are two equally likely outcomes: $\{T, H\}$. The probability is 1/2.

(b) There are four equally likely outcomes: $\{TT, TH, HT, HH\}$. The probability is 1/4.

(c) There are 52 equally likely outcomes, 4 of which are queens. The probability is 4/52, or 1/13.

(d) By the Multiplication Principle of Counting (Section 9.1), there are $6 \times 6 = 36$ equally likely outcomes. Of these, three $\{(1, 3), (3, 1), (2, 2)\}$ yield a sum of 4 (Figure 9.2). The probability is 3/36, or 1/12.

(e) There are ${}_{46}C_6 = 9{,}366{,}819$ equally likely ways that 6 numbers can be chosen from 46 numbers without regard to order. Only one of these choices wins the lottery. The probability is $1/9{,}366{,}819 \approx 0.00000010676$. *Now try Exercise 5.*

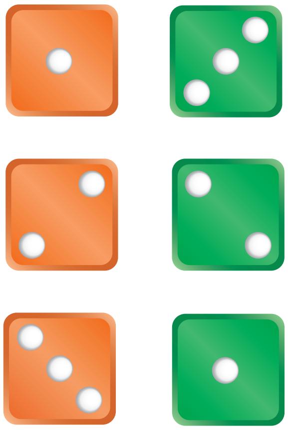

FIGURE 9.2 A sum of 4 on a roll of two dice. (Example 1d)

Notice that in each of these cases we first counted the number of possible outcomes of the experiment in question. The set of all possible outcomes of an experiment is the **sample space** of the experiment. An **event** is a subset of the sample space. Each of our sample spaces consisted of a finite number of **equally likely outcomes**, which enabled us to find the probability of an event by counting.

Probability of an Event (Equally Likely Outcomes)

If E is an event in a finite, nonempty sample space S of equally likely outcomes, then the **probability** of the event E is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the number of outcomes in } S}.$$

The hypothesis of equally likely outcomes is critical here. Many people guess wrongly on the probability in Example 1d because they figure that there are 11 possible outcomes for the sum on two fair dice: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and that 4 is one of them. (That reasoning is correct so far.) The reason that 1/11 is not the probability of rolling a sum of 4 is that all those sums are *not equally likely*.

Is Probability Just for Games?

Probability theory got its start in letters between Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665) concerning games of chance, but it has come a long way since then. Modern mathematicians like David Blackwell (1919), the first African-American to receive a fellowship to the Institute for Advanced Study at Princeton, have greatly extended both the theory and the applications of probability, especially in the areas of statistics, quantum physics, and information theory. Moreover, the work of John Von Neumann (1903–1957) has led to a separate branch of modern discrete mathematics that really is about games, called game theory.

On the other hand, we can *assign* probabilities to the 11 outcomes in this smaller sample space in a way that is consistent with the number of ways each total can occur. The table below shows a **probability distribution**, in which each outcome is assigned a unique probability by a *probability function*.

Outcome	Probability
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

We see that the outcomes are not equally likely, but we can find the probabilities of events by adding up the probabilities of the outcomes in the event, as in the following example.

EXAMPLE 2 Rolling the Dice

Find the probability of rolling a sum divisible by 3 on a single roll of two fair dice.

SOLUTION The event E consists of the outcomes $\{3, 6, 9, 12\}$. To get the probability of E we add up the probabilities of the outcomes in E (see the table of the probability distribution):

$$P(E) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{12}{36} = \frac{1}{3}.$$

Now try Exercise 7.

Notice that this method would also have worked just fine with our 36-outcome sample space, in which every outcome has probability 1/36. In general, it is easier to work with sample spaces of equally likely events because it is not necessary to write out the probability distribution. When outcomes do have unequal probabilities, we need to know what probabilities to assign to the outcomes.

Not every function that assigns numbers to outcomes qualifies as a probability function.

Empty Set

A set with no elements is the *empty set*, denoted by $\varnothing$.

DEFINITION Probability Function

A **probability function** is a function P that assigns a real number to each outcome in a sample space S subject to the following conditions:

1. $0 \le P(O) \le 1$ for every outcome O;

2. the sum of the probabilities of all outcomes in S is 1;

3. $P(\varnothing) = 0$.

Random Variables

A more formal treatment of probability would distinguish between an *outcome* in a sample space and a *number* that is associated with that outcome. For example, an outcome in Example 2 is really something like , to which we associate the number $2 + 1 = 3$. A function that assigns a number to an outcome is called a **random variable**. A different random variable might assign the number $2 \cdot 1 = 2$ to this outcome of the dice. Another random variable might assign the number 21.

The probability of any event can then be defined in terms of the probability function.

Probability of an Event (Outcomes Not Equally Likely)

Let S be a finite, nonempty sample space in which every outcome has a probability assigned to it by a probability function P. If E is any event in S, the **probability** of the event E is the sum of the probabilities of all the outcomes contained in E.

EXAMPLE 3 Testing a Probability Function

Is it possible to weight a standard 6-sided die in such a way that the probability of rolling each number n is exactly $1/(n^2 + 1)$?

SOLUTION The probability distribution would look like this:

Outcome	Probability
1	1/2
2	1/5
3	1/10
4	1/17
5	1/26
6	1/37

This is not a valid probability function, because $1/2 + 1/5 + 1/10 + 1/17 + 1/26 + 1/37 \neq 1$. *Now try Exercise 9a.*

Determining Probabilities

It is not always easy to determine probabilities, but the arithmetic involved is fairly simple. It usually comes down to multiplication, addition, and (most importantly) counting. Here is the strategy we will follow:

Strategy for Determining Probabilities

1. Determine the sample space of all possible outcomes. When possible, choose outcomes that are equally likely.
2. If the sample space has equally likely outcomes, the probability of an event E is determined by counting:

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the number of outcomes in } S}$$

3. If the sample space does not have equally likely outcomes, determine the probability function. (This is not always easy to do.) Check to be sure that the conditions of a probability function are satisfied. Then the probability of an event E is determined by adding up the probabilities of all the outcomes contained in E.

EXAMPLE 4 Choosing Chocolates, Sample Space I

Sal opens a box of a dozen chocolate cremes and generously offers two of them to Val. Val likes vanilla cremes the best, but all the chocolates look alike on the outside. If four of the twelve cremes are vanilla, what is the probability that both of Val's picks turn out to be vanilla?

SOLUTION The experiment in question is the selection of two chocolates, without regard to order, from a box of 12. There are ${}_{12}C_2 = 66$ outcomes of this experiment, and all of them are equally likely. We can therefore determine the probability by counting.

The event E consists of all possible pairs of 2 vanilla cremes that can be chosen, without regard to order, from 4 vanilla cremes available. There are ${}_4C_2 = 6$ ways to form such pairs.

Therefore, $P(E) = 6/66 = 1/11$. *Now try Exercise 25.*

Many probability problems require that we think of events happening in succession, often with the occurrence of one event affecting the probability of the occurrence of another event. In these cases, we use a law of probability called the Multiplication Principle of Probability.

Multiplication Principle of Probability

Suppose an event A has probability p_1 and an event B has probability p_2 under the assumption that A occurs. Then the probability that both A and B occur is $p_1 p_2$.

If the events A and B are **independent**, we can omit the phrase "under the assumption that A occurs," since that assumption would not matter.

As an example of this principle at work, we will solve the *same problem* as that posed in Example 4, this time using a sample space that appears at first to be simpler, but which consists of events that are not equally likely.

EXAMPLE 5 Choosing Chocolates, Sample Space II

Sal opens a box of a dozen chocolate cremes and generously offers two of them to Val. Val likes vanilla cremes the best, but all the chocolates look alike on the outside. If four of the twelve cremes are vanilla, what is the probability that both of Val's picks turn out to be vanilla?

SOLUTION As far as Val is concerned, there are two kinds of chocolate cremes: vanilla (V) and unvanilla (U). When choosing two chocolates, there are four possible outcomes: VV, VU, UV, and UU. We need to determine the probability of the outcome VV.

Notice that these four outcomes are *not equally likely*! There are twice as many U chocolates as V chocolates. So we need to consider the distribution of probabilities, and we may as well begin with $P(VV)$, as that is the probability we seek.

The probability of picking a vanilla creme on the first draw is 4/12. The probability of picking a vanilla creme on the second draw, *under the assumption that a vanilla creme was drawn on the first*, is 3/11. By the Multiplication Principle, the probability of drawing a vanilla creme on both draws is

$$\frac{4}{12} \cdot \frac{3}{11} = \frac{1}{11}.$$

Since this is the probability we are looking for, we do not need to compute the probabilities of the other outcomes. However, you should verify that the other probabilities would be:

$$P(VU) = \frac{4}{12} \cdot \frac{8}{11} = \frac{8}{33}$$

$$P(UV) = \frac{8}{12} \cdot \frac{4}{11} = \frac{8}{33}$$

$$P(UU) = \frac{8}{12} \cdot \frac{7}{11} = \frac{14}{33}$$

Notice that $P(VV) + P(VU) + P(UV) + P(UU) = (1/11) + (8/33) + (8/33) + (14/33) = 1$, so the probability function checks out. *Now try Exercise 33.*

Ordered or Unordered?

Notice that in Example 4 we had a sample space in which order was disregarded, whereas in Example 5 we have a sample space in which order matters. (For example, UV and VU are distinct outcomes.) The order matters in Example 5 because we are considering the probabilities of two events (first draw, second draw), one of which affects the other. In Example 4, we are simply counting unordered combinations.

Venn Diagrams and Tree Diagrams

We have seen many instances in which geometric models help us to understand algebraic models more easily, and probability theory is yet another setting in which this is true. **Venn diagrams**, associated mainly with the world of set theory, are good for visualizing

John Venn

John Venn (1834–1923) was an English logician and clergyman, just like his contemporary, Charles L. Dodgson. Although both men used overlapping circles to illustrate their logical syllogisms, it is Venn whose name lives on in connection with these diagrams. Dodgson's name barely lives on at all, and yet he is far the more famous of the two: under the pen name Lewis Carroll, he wrote *Alice's Adventures in Wonderland* and *Through the Looking Glass*.

Addition Principle of Probability

A more careful look at the Venn diagram in Figure 9.4 will suggest the following general formula for events A and B in a sample space:

$$P(A \text{ or } B) = P(A) + (B) - P(A \text{ and } B)$$

If A and B do not happen to intersect, this reduces to $P(A \text{ or } B) = P(A) + (B)$. In this case, we call A and B **mutually exclusive** events.

relationships among events within sample spaces. **Tree diagrams**, which we first met in Section 9.1 as a way to visualize the Multiplication Principle of Counting, are good for visualizing the Multiplication Principle of Probability.

EXAMPLE 6 Using a Venn Diagram

In a large high school, 54% of the students are girls and 62% of the students play sports. Half of the girls at the school play sports.

(a) What percentage of the students who play sports are boys?

(b) If a student is chosen at random, what is the probability that it is a boy who does not play sports?

SOLUTION To organize the categories, we draw a large rectangle to represent the sample space (all students at the school) and two overlapping regions to represent "girls" and "sports" (Figure 9.3). We fill in the percentages (Figure 9.4) using the following logic:

- The overlapping (green) region contains half the girls, or $(0.5)(54\%) = 27\%$ of the students.
- The yellow region (the rest of the girls) then contains $(54 - 27)\% = 27\%$ of the students.
- The blue region (the rest of the sports players) then contains $(62 - 27)\% = 35\%$ of the students.
- The white region (the rest of the students) then contains $(100 - 89)\% = 11\%$ of the students. These are boys who do not play sports.

We can now answer the two questions by looking at the Venn diagram.

(a) We see from the diagram that the ratio of *boys* who play sports to *all students* who play sports is $\frac{0.35}{0.62}$, which is about 56.45%.

(b) We see that 11% of the students are boys who do not play sports, so 0.11 is the probability. *Now try Exercises 27a–d.*

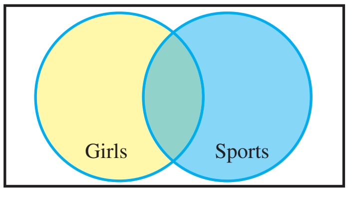

FIGURE 9.3 A Venn diagram for Example 6. The overlapping region common to both circles represents "girls who play sports." The region outside both circles (but inside the rectangle) represents "boys who do not play sports."

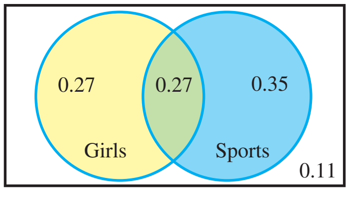

FIGURE 9.4 A Venn diagram for Example 6 with the probabilities filled in.

EXAMPLE 7 Using a Tree Diagram

Two identical cookie jars are on a counter. Jar A contains 2 chocolate chip and 2 peanut butter cookies, while jar B contains 1 chocolate chip cookie. We select a cookie at random. What is the probability that it is a chocolate chip cookie?

SOLUTION It is tempting to say 3/5, since there are 5 cookies in all, 3 of which are chocolate chip. Indeed, this would be the answer if all the cookies were in the same jar. However, the fact that they are in different jars means that the 5 cookies are *not equally likely outcomes*. That lone chocolate chip cookie in jar B has a much

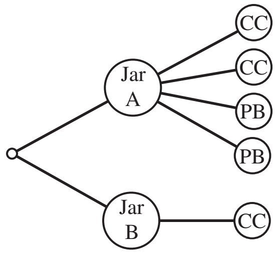

FIGURE 9.5 A tree diagram for Example 7.

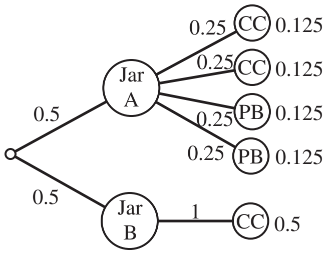

FIGURE 9.6 The tree diagram for Example 7 with the probabilities filled in. Notice that the five cookies are not equally likely to be drawn. Notice also that the probabilities of the five cookies do add up to 1.

better chance of being chosen than any of the cookies in jar A. We need to think of this as a two-step experiment: first pick a jar, then pick a cookie from that jar.

Figure 9.5 gives a visualization of the two-step process. In Figure 9.6, we have filled in the probabilities along each branch, first of picking the jar, then of picking the cookie. The probability at the *end* of each branch is obtained by multiplying the probabilities from the root to the branch. (This is the Multiplication Principle.) Notice that the probabilities of the 5 cookies (as predicted) are not equal.

The event "chocolate chip" is a set containing three outcomes. We add their probabilities together to get the correct probability:

$P(\text{chocolate chip}) = 0.125 + 0.125 + 0.5 = 0.75$ *Now try Exercise 29.*

Conditional Probability

The probability of drawing a chocolate chip cookie in Example 7 is an example of **conditional probability**, since the "cookie" probability is **dependent** on the "jar" outcome. A convenient symbol to use with conditional probability is $P(A|B)$, pronounced "P of A given B," meaning "the probability of the event A, given that event B occurs." In the cookie jars of Example 7,

$$P(\text{chocolate chip} \mid \text{jar } A) = \frac{2}{4} \quad \text{and} \quad P(\text{chocolate chip} \mid \text{jar } B) = 1$$

(In the tree diagram, these are the probabilities along the *branches* that come out of the two jars, not the probabilities at the *ends* of the branches.)

The Multiplication Principle of Probability can be stated succinctly with this notation as follows:

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

This is how we found the numbers at the ends of the branches in Figure 9.6.

As our final example of a probability problem, we will show how to use this formula in a different but equivalent form, sometimes called the **conditional probability formula**:

Conditional Probability Formula

If the event B depends on the event A, then $P(B \mid A) = \dfrac{P(A \text{ and } B)}{P(A)}$.

EXAMPLE 8 Using the Conditional Probability Formula

Suppose we have drawn a cookie at random from one of the jars described in Example 7. Given that it is chocolate chip, what is the probability that it came from jar A?

SOLUTION By the formula,

$$P(\text{jar } A \mid \text{chocolate chip}) = \frac{P(\text{jar } A \text{ and chocolate chip})}{P(\text{chocolate chip})}$$

$$= \frac{(1/2)(2/4)}{0.75} = \frac{0.25}{0.75} = \frac{1}{3}$$

Now try Exercise 31.

EXPLORATION 1 Testing Positive for HIV

As of the year 2003, the probability of an adult in the United States having HIV/AIDS was 0.006 (source: *2004 CIA World Factbook*). The ELISA test is used to detect the virus antibody in blood. If the antibody is present, the test reports positive with probability 0.997 and negative with probability 0.003. If the antibody is not present, the test reports positive with probability 0.015 and negative with probability 0.985.

1. Draw a tree diagram with branches to nodes "antibody present" and "antibody absent" branching from the root. Fill in the probabilities for North American adults (age 15–49) along the branches. (Note that these two probabilities must add up to 1.)
2. From the node at the end of each of the two branches, draw branches to "positive" and "negative." Fill in the probabilities along the branches.
3. Use the Multiplication Principle to fill in the probabilities at the ends of the four branches. Check to see that they add up to 1.
4. Find the probability of a positive test result. (Note that this event consists of two outcomes.)
5. Use the conditional probability formula to find the probability that a person with a positive test result actually *has* the antibody, i.e., $P(\text{antibody present} \mid \text{positive})$.

You might be surprised that the answer to part 5 is so low, but it should be compared with the probability of the antibody being present *before* seeing the positive test result, which was 0.006. Nonetheless, that is why a positive ELISA test is followed by further testing before a diagnosis of HIV/AIDS is made. This is the case with many diagnostic tests.

Binomial Distributions

We noted in our "Strategy for Determining Probabilities" (page 660) that it is not always easy to determine a probability distribution for a sample space with unequal probabilities. An interesting exception for those who have studied the Binomial Theorem (Section 9.2) is the binomial distribution.

Talking the Talk

Notice that the probabilities in Example 9 are associated with the random variable that *counts the number of 3's* on four rolls of a fair die. For this random variable, then, $P(4) \approx 0.00077$, $P(0) \approx 0.48225$, and $P(2) \approx 0.01929$. (See the margin note on **random variables** on page 659.)

EXAMPLE 9 Repeating a Simple Experiment

We roll a fair die four times. Find the probability that we roll:

(a) All 3's. **(b)** No 3's. **(c)** Exactly two 3's.

SOLUTION

(a) We have a probability 1/6 of rolling a 3 each time. By the Multiplication Principle, the probability of rolling a 3 all four times is $(1/6)^4 \approx 0.00077$.

(b) There is a probability 5/6 of rolling something other than 3 each time. By the Multiplication Principle, the probability of rolling a non-3 all four times is $(5/6)^4 \approx 0.48225$.

(c) The probability of rolling two 3's followed by two non-3's (again by the Multiplication Principle) is $(1/6)^2(5/6)^2 \approx 0.01929$. However, that is not the only outcome we must consider. In fact, the two 3's could occur *anywhere* among the four rolls, in exactly $\binom{4}{2} = 6$ ways. That gives us 6 outcomes, each with probability $(1/6)^2(5/6)^2$. The probability of the event "exactly two 3's" is therefore $\binom{4}{2}(1/6)^2(5/6)^2 \approx 0.11574$.

Now try Exercise 47.

Did the form of those answers look a little familiar? Watch what they look like when we let $p = 1/6$ and $q = 5/6$:

$$P(\text{four 3's}) = p^4$$
$$P(\text{no 3's}) = q^4$$
$$P(\text{two 3's}) = \binom{4}{2}p^2q^2$$

You can probably recognize these as three of the terms in the expansion of $(p + q)^4$. This is no coincidence. In fact, the terms in the expansion

$$(p + q)^4 = p^4 + 4p^3q^1 + 6p^2q^2 + 4p^1q^3 + q^4$$

give the exact probabilities of 4, 3, 2, 1, and 0 threes (respectively) when we toss a fair die four times! That is why this is called a *binomial probability distribution*. The general theorem follows.

Binomial Probabilities on a Calculator

Your calculator might be programmed to find values for the binomial probability distribution function (binompdf). The solutions to Example 10 in one calculator syntax, for example, could be obtained by:

(a) binompdf(20, .9, 20) (20 repetitions, 0.9 probability, 20 successes)

(b) binompdf(20, .9, 18) (20 repetitions, 0.9 probability, 18 successes)

(c) 1 − binomcdf(20, .9, 17) (1 minus the *cumulative* probability of 17 or fewer successes)

Check your owner's manual for more information.

THEOREM Binomial Distribution

Suppose an experiment consists of n independent repetitions of an experiment with two outcomes, called "success" and "failure." Let $P(\text{success}) = p$ and $P(\text{failure}) = q$. (Note that $q = 1 - p$.)

Then the terms in the binomial expansion of $(p + q)^n$ give the respective probabilities of exactly $n, n - 1, \ldots, 2, 1, 0$ successes. The distribution is shown below:

Number of successes out of n independent repetitions	Probability
n	p^n
$n - 1$	$\binom{n}{n-1}p^{n-1}q$
$\vdots$	$\vdots$
r	$\binom{n}{r}p^r q^{n-r}$
$\vdots$	$\vdots$
1	$\binom{n}{1}pq^{n-1}$
0	q^n

EXAMPLE 10 Shooting Free Throws

Suppose Michael makes 90% of his free throws. If he shoots 20 free throws, and if his chance of making each one is independent of the other shots (an assumption you might question in a game situation), what is the probability that he makes

(a) All 20? **(b)** Exactly 18? **(c)** At least 18?

SOLUTION We could get the probabilities of all possible outcomes by expanding $(0.9 + 0.1)^{20}$, but that is not necessary in order to answer these three questions. We just need to compute three specific terms.

(a) $P(20 \text{ successes}) = (0.9)^{20} \approx 0.12158$

(b) $P(18 \text{ successes}) = \binom{20}{18}(0.9)^{18}(0.1)^2 \approx 0.28518$

(c)
$$\begin{aligned} P(\text{at least 18 successes}) &= P(18) + P(19) + P(20) \\ &= \binom{20}{18}(0.9)^{18}(0.1)^2 + \binom{20}{19}(0.9)^{19}(0.1) + (0.9)^{20} \\ &\approx 0.6769 \end{aligned}$$

Now try Exercise 49.

QUICK REVIEW 9.3 *(Prerequisite skill Section 9.1)*

In Exercises 1–8, tell how many outcomes are possible for the experiment.

1. A single coin is tossed.

2. A single 6-sided die is rolled.

3. Three different coins are tossed.

4. Three different 6-sided dice are rolled.

5. Five different cards are drawn from a standard deck of 52.

6. Two chips are drawn simultaneously from a jar containing 10 chips.

7. Five people are lined up for a photograph.

8. Three-digit numbers are formed from the numbers $\{1, 2, 3, 4, 5\}$ without repetition.

In Exercises 9 and 10, evaluate the expression by pencil and paper. Verify your answer with a calculator.

9. $\dfrac{{}_5C_3}{{}_{10}C_3}$ **10.** $\dfrac{{}_5C_2}{{}_{10}C_2}$

SECTION 9.3 EXERCISES

In Exercises 1–8, a red die and a green die have been rolled. What is the probability of the event?

1. The sum is 9. **2.** The sum is even.

3. The number on the red die is greater than the number on the green die.

4. The sum is less than 10.

5. Both numbers are odd.

6. Both numbers are even.

7. The sum is prime. **8.** The sum is 7 or 11.

9. Writing to Learn Alrik's gerbil cage has four compartments, A, B, C, and D. After careful observation, he estimates the proportion of time the gerbil spends in each compartment and constructs the table below.

Compartment	A	B	C	D
Proportion	0.25	0.20	0.35	0.30

(a) Is this a valid probability function? Explain.

(b) Is there a problem with Alrik's reasoning? Explain.

10. (Continuation of Exercise 9) Suppose Alrik determines that his gerbil spends time in the four compartments A, B, C, and D in the ratio 4:3:2:1. What proportions should he fill in the table above? Is this a valid probability function?

The maker of a popular chocolate candy that is covered in a thin colored shell has released information about the overall color proportions in its production of the candy, which is summarized in the following table.

Color	Brown	Red	Yellow	Green	Orange	Tan
Proportion	0.3	0.2	0.2	0.1	0.1	0.1

In Exercises 11–16, a single candy of this type is selected at random from a newly opened bag. What is the probability that the candy has the given color(s)?

11. Brown or tan **12.** Red, green, or orange

13. Red **14.** Not red

15. Neither orange nor yellow **16.** Neither brown nor tan

A peanut version of the same candy has all the same colors except tan. The proportions of the peanut version are given in the following table.

Color	Brown	Red	Yellow	Green	Orange
Proportion	0.3	0.2	0.2	0.2	0.1

In Exercises 17–22, a candy of this type is selected at random from each of two newly opened bags. What is the probability that the two candies have the given color(s)?

17. Both are brown.

18. Both are orange.

19. One is red, and the other is green.

20. The first is brown, and the second is yellow.

21. Neither is yellow.

22. The first is not red, and the second is not orange.

Exercises 23–32 concern a version of the card game "bid Euchre" that uses a pack of 24 cards, consisting of ace, king, queen, jack, 10, and 9 in each of the four suits (spades, hearts, diamonds, and clubs). In bid Euchre, a hand consists of 6 cards. Find the probability of each event.

23. Euchre A hand is all spades.

24. Euchre All six cards are from the same suit.

25. Euchre A hand includes all four aces.

26. Euchre A hand includes two jacks of the same color (called the right and left bower).

27. Using Venn Diagrams A and B are events in a sample space S such that $P(A) = 0.6$, $P(B) = 0.5$, and $P(A \text{ and } B) = 0.3$.

(a) Draw a Venn diagram showing the overlapping sets A and B and fill in the probabilities of the four regions formed.

(b) Find the probability that A occurs but B does not.

(c) Find the probability that B occurs but A does not.

(d) Find the probability that neither A nor B occurs.

(e) Are events A and B independent? (That is, does $P(A \mid B) = P(A)$?)

28. Using Venn Diagrams A and B are events in a sample space S such that $P(A) = 0.7$, $P(B) = 0.4$, and $P(A \text{ and } B) = 0.2$.

(a) Draw a Venn diagram showing the overlapping sets A and B and fill in the probabilities of the four regions formed.

(b) Find the probability that A occurs but B does not.

(c) Find the probability that B occurs but A does not.

(d) Find the probability that neither A nor B occurs.

(e) Are events A and B independent? (That is, does $P(A \mid B) = P(A)$?)

In Exercises 29 and 30, it will help to draw a tree diagram.

29. Piano Lessons If it rains tomorrow, the probability is 0.8 that John will practice his piano lesson. If it does not rain tomorrow, there is only a 0.4 chance that John will practice. Suppose that the chance of rain tomorrow is 60%. What is the probability that John will practice his piano lesson?

30. Predicting Cafeteria Food If the school cafeteria serves meat loaf, there is a 70% chance that they will serve peas. If they do not serve meat loaf, there is a 30% chance that they will serve peas anyway. The students know that meat loaf will be served exactly once during the 5-day week, but they do not know which day. If tomorrow is Monday, what is the probability that

(a) The cafeteria serves meat loaf?

(b) The cafeteria serves meat loaf and peas?

(c) The cafeteria serves peas?

31. Conditional Probability There are two precalculus sections at West High School. Mr. Abel's class has 12 girls and 8 boys, while Mr. Bonitz's class has 10 girls and 15 boys. If a West High precalculus student chosen at random happens to be a girl, what is the probability she is from Mr. Abel's class? [*Hint*: The answer is not 12/22.]

32. Group Activity Conditional Probability Two boxes are on the table. One box contains a normal coin and a two-headed coin; the other box contains three normal coins. A friend reaches into a box, removes a coin, and shows you one side: a head. What is the probability that it came from the box with the two-headed coin?

33. Renting Cars Floppy Jalopy Rent-a-Car has 25 cars available for rental—20 big bombs and 5 midsize cars. If two cars are selected at random, what is the probability that both are big bombs?

34. Defective Calculators Dull Calculators, Inc., knows that a unit coming off an assembly line has a probability of 0.037 of being defective. If four units are selected at random during the course of a workday, what is the probability that none of the units are defective?

35. Causes of Death The government designates a single cause for each death in the United States. The resulting data indicate that 45% of deaths are due to heart and other cardiovascular disease and 22% are due to cancer.

(a) What is the probability that the death of a randomly selected person will be due to cardiovascular disease or cancer?

(b) What is the probability that the death will be due to some other cause?

36. Yahtzee In the game of *Yahtzee*, on the first roll five dice are tossed simultaneously. What is the probability of rolling five of a kind (which is Yahtzee!) on the first roll?

37. Writing to Learn Explain why the following statement cannot be true. The probabilities that a computer salesperson will sell zero, one, two, or three computers in any one day are 0.12, 0.45, 0.38, and 0.15, respectively.

38. HIV Testing A particular test for HIV, the virus that causes AIDS, is 0.7% likely to produce a false positive result—a result indicating that the human subject has HIV when in fact the person is not carrying the virus. If 60 individuals who are HIV-negative are tested, what is the probability of obtaining at least one false result?

39. Graduate School Survey The Earmuff Junction College Alumni Office surveys selected members of the class of 2000. Of the 254 who graduated that year, 172 were women, 124 of whom went on to graduate school. Of the male graduates, 58 went on to graduate school. What is the probability of the given event?

(a) The graduate is a woman.

(b) The graduate went on to graduate school.

(c) The graduate was a woman who went on to graduate school.

40. Indiana Jones and the Final Exam Professor Indiana Jones gives his class a list of 20 study questions, from which he will select 8 to be answered on the final exam. If a given student knows how to answer 14 of the questions, what is the probability that the student will be able to answer the given number of questions correctly?

(a) All 8 questions

(b) Exactly 5 questions

(c) At least 6 questions

41. Graduation Requirement To complete the kinesiology requirement at Palpitation Tech you must pass two classes chosen from aerobics, aquatics, defense arts, gymnastics, racket sports, recreational activities, rhythmic activities, soccer, and volleyball. If you decide to choose your two classes at random by drawing two class names from a box, what is the probability that you will take racket sports and rhythmic activities?

42. Writing to Learn During July in Gunnison, Colorado, the probability of at least 1 hour a day of sunshine is 0.78, the probability of at least 30 minutes of rain is 0.44, and the probability that it will be cloudy all day is 0.22. Write a paragraph explaining whether this statement could be true.

In Exercises 43–50, ten dimes dated 1990 through 1999 are tossed. Find the probability of each event.

43. Tossing Ten Dimes Heads on the 1990 dime only

44. Tossing Ten Dimes Heads on the 1991 and 1996 dimes only

45. Tossing Ten Dimes Heads on all 10 dimes

46. Tossing Ten Dimes Heads on all but one dime

47. Tossing Ten Dimes Exactly two heads

48. Tossing Ten Dimes Exactly three heads

49. Tossing Ten Dimes At least one head

50. Tossing Ten Dimes At least two heads

Standardized Test Questions

51. True or False A sample space consists of equally likely events. Justify your answer.

52. True or False The probability of an event can be greater than 1. Justify your answer.

Evaluate Exercises 53–56 without using a calculator.

53. Multiple Choice The probability of rolling a total of 5 on a pair of fair dice is

(A) $\frac{1}{4}$. **(B)** $\frac{1}{5}$.

(C) $\frac{1}{6}$. **(D)** $\frac{1}{9}$.

(E) $\frac{1}{11}$.

54. Multiple Choice Which of the following numbers could not be the probability of an event?

(A) 0 **(B)** 0.95

(C) $\frac{\sqrt{3}}{4}$ **(D)** $\frac{3}{\pi}$

(E) $\frac{\pi}{2}$

55. Multiple Choice If A and B are independent events, then $P(A \mid B) =$

(A) $P(A)$. **(B)** $P(B)$.

(C) $P(B \mid A)$. **(D)** $P(A) \cdot P(B)$.

(E) $P(A) + P(B)$.

56. Multiple Choice A fair coin is tossed three times in succession. What is the probability that exactly one of the coins shows heads?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{3}$

(C) $\frac{3}{8}$ **(D)** $\frac{1}{2}$

(E) $\frac{2}{3}$

Explorations

57. Empirical Probability In real applications, it is often necessary to approximate the probabilities of the various outcomes of an experiment by performing the experiment a large number of times and recording the results. Barney's Bread Basket offers five different kinds of bagels. Barney records the sales of the first 500 bagels in a given week in the table shown below:

Type of Bagel	Number Sold
Plain	185
Onion	60
Rye	55
Cinnamon Raisin	125
Sourdough	75

(a) Use the observed sales number to approximate the probability that a random customer buys a plain bagel. Do the same for each other bagel type and make a table showing the approximate probability distribution.

(b) Assuming independence of the events, find the probability that three customers in a row all order plain bagels.

(c) Writing to Learn Do you think it is reasonable to assume that the orders of three consecutive customers actually are independent? Explain.

58. Straight Poker In the original version of poker known as "straight" poker, a 5-card hand is dealt from a standard deck of 52 cards. What is the probability of the given event?

(a) A hand will contain at least one king.

(b) A hand will be a "full house" (any three of one kind and a pair of another kind).

59. Married Students Suppose that 23% of all college students are married. Answer the following questions for a random sample of eight college students.

(a) How many would you expect to be married?

(b) Would you regard it as unusual if the sample contained five married students?

(c) What is the probability that five or more of the eight students are married?

60. Group Activity Investigating an Athletic Program A university widely known for its track and field program claims that 75% of its track athletes get degrees. A journalist investigates what happened to the 32 athletes who began the program over a 6-year period that ended 7 years ago. Of these athletes, 17 have graduated and the remaining 15 are no longer attending any college. If the university's claim is true, the number of athletes who graduate among the 32 examined should have been governed by binomial probability with $p = 0.75$.

(a) What is the probability that exactly 17 athletes should have graduated?

(b) What is the probability that 17 or fewer athletes should have graduated?

(c) If you were the journalist, what would you say in your story on the investigation?

Extending the Ideas

61. Expected Value If the outcomes of an experiment are given numerical values (such as the total on a roll of two dice, or the payoff on a lottery ticket), we define the **expected value** to be the sum of all the numerical values times their respective probabilities.

For example, suppose we roll a fair die. If we roll a multiple of 3, we win \$3; otherwise we lose \$1. The probabilities of the two possible payoffs are shown in the table below:

Value	Probability
$+3$	2/6
-1	4/6

The expected value is

$$3 \times (2/6) + (-1) \times (4/6) = (6/6) - (4/6) = 1/3.$$

We interpret this to mean that we would win an average of 1/3 dollar per game in the long run.

(a) A game is called *fair* if the expected value of the payoff is zero. Assuming that we still win \$3 for a multiple of 3, what should we pay for any other outcome in order to make the game fair?

(b) Suppose we roll *two* fair dice and look at the total under the original rules. That is, we win \$3 for rolling a multiple of 3 and lose \$1 otherwise. What is the expected value of this game?

62. Expected Value (Continuation of Exercise 61) Gladys has a personal rule never to enter the lottery (picking 6 numbers from 1 to 46) until the payoff reaches 4 million dollars. When it does reach 4 million, she always buys ten different \$1 tickets.

(a) Assume that the payoff for a winning ticket is 4 million dollars. What is the probability that Gladys holds a winning ticket? (Refer to Example 1 of this section for the probability of any ticket winning.)

(b) Fill in the probability distribution for Gladys's possible payoffs in the table below. (Note that we subtract \$10 from the \$4 million, since Gladys has to pay for her tickets even if she wins.)

Value	Probability
-10	
$+3{,}999{,}990$	

(c) Find the expected value of the game for Gladys.

(d) **Writing to Learn** In terms of the answer in part (b), explain to Gladys the long-term implications of her strategy.