CHAPTER 9

Discrete Mathematics

As the use of cellular telephones, modems, pagers, and fax machines has grown in recent years, the increasing demand for unique telephone numbers has necessitated the creation of new area codes in many areas of the United States. Counting the number of possible telephone numbers in a given area code is a *combinatorial* problem, and such problems are solved using the techniques of *discrete* mathematics. See page 648 for more on the subject of telephone area codes.

Chapter 9 Overview

The branches of mathematics known broadly as algebra, analysis, and geometry come together so beautifully in calculus that it has been difficult over the years to squeeze other mathematics into the curriculum. Consequently, many worthwhile topics like probability and statistics, combinatorics, graph theory, and numerical analysis that could easily be introduced in high school are (for many students) either first seen in college electives or never seen at all. This situation is gradually changing as the applications of noncalculus mathematics become increasingly more important in the modern, computerized, data-driven workplace. Therefore, besides introducing important topics like sequences and series and the Binomial Theorem, this chapter will touch on some other discrete topics that might prove useful to you in the near future.

9.1 Basic Combinatorics

What you'll learn about

- Discrete Versus Continuous
- The Importance of Counting
- The Multiplication Principle of Counting
- Permutations
- Combinations
- Subsets of an n-Set

... and why

Counting large sets is easy if you know the correct formula.

Discrete Versus Continuous

A point has no length and no width, and yet intervals on the real line—which are made up of these dimensionless points—have length! This little mystery illustrates the distinction between *continuous* and *discrete* mathematics. Any interval (a, b) contains a **continuum** of real numbers, which is why you can zoom in on an interval forever and there will still be an interval there. Calculus concepts like limits and continuity depend on the mathematics of the continuum. In *discrete* mathematics, we are concerned with properties of numbers and algebraic systems that do not depend on that kind of analysis. Many of them are related to the first kind of mathematics that most of us ever did, namely counting. Counting is what we will do for the rest of this section.

The Importance of Counting

We begin with a relatively simple counting problem.

EXAMPLE 1 Arranging Three Objects in Order

In how many different ways can three distinguishable objects be arranged in order?

SOLUTION It is not difficult to list all the possibilities. If we call the objects *A, B,* and *C*, the different orderings are: *ABC, ACB, BAC, BCA, CAB,* and *CBA*. A good way to visualize the six choices is with a *tree diagram*, as in Figure 9.1. Notice that we have three choices for the first letter. Then, branching off each of those three choices are two choices for the second letter. Finally, branching off each of the $3 \times 2 = 6$ branches formed so far is one choice for the third letter. By beginning at the "root" of the tree, we can proceed to the right along any of the $3 \times 2 \times 1 = 6$ branches and get a different ordering each time. We conclude that there are six ways to arrange three distinguishable objects in order. ***Now try Exercise 3.***

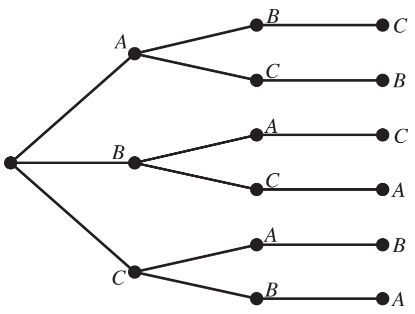


FIGURE 9.1 A tree diagram for ordering the letters *ABC*. (Example 1)

Scientific studies will usually manipulate one or more **explanatory** variables and observe the effect of that manipulation on one or more **response** variables. The key to understanding the significance of the effect is to know what is likely to occur *by chance alone*, and that often depends on counting. For example, Exploration 1 shows a real-world application of Example 1.

EXPLORATION 1 Questionable Product Claims

A salesman for a copying machine company is trying to convince a client to buy his $2000 machine instead of his competitor's $5000 machine. To make his point, he lines up an original document, a copy made by his machine, and a copy made by the more expensive machine on a table and asks 60 office workers to identify which is which. To everyone's surprise, not a single worker identifies all three correctly. The salesman states triumphantly that this proves that all three documents look the same to the naked eye and that therefore the client should buy his company's less expensive machine.

What do you think?

1. Each worker is essentially being asked to put the three documents in the correct order. How many ways can the three documents be ordered?
2. Suppose all three documents really *do* look alike. What fraction of the workers would you expect to put them into the correct order by chance alone?
3. If zero people out of 60 put the documents in the correct order, should we conclude that "all three documents look the same to the naked eye"?
4. Can you suggest a more likely conclusion that we might draw from the results of the salesman's experiment?

The Multiplication Principle of Counting

You would not want to draw the tree diagram for ordering five objects (*ABCDE*), but you should be able to see in your mind that it would have $5 \times 4 \times 3 \times 2 \times 1 = 120$ branches. A tree diagram is a geometric visualization of a fundamental counting principle known as the *Multiplication Principle*.

Multiplication Principle of Counting

If a procedure P has a sequence of stages $S_1, S_2, \ldots, S_n$ and if

$$\begin{aligned} &S_1 \text{ can occur in } r_1 \text{ ways,}\\ &S_2 \text{ can occur in } r_2 \text{ ways,}\\ &\vdots\\ &S_n \text{ can occur in } r_n \text{ ways,}\end{aligned}$$

then the number of ways that the procedure P can occur is the product

$$r_1 r_2 \cdots r_n.$$

It is important to be mindful of how the choices at each stage are affected by the choices at preceding stages. For example, when choosing an order for the letters *ABC* we have 3 choices for the first letter, but only 2 choices for the second and 1 for the third.

EXAMPLE 2 Using the Multiplication Principle

The Tennessee license plate shown here consists of three letters of the alphabet followed by three numerical digits (0 through 9). Find the number of different license plates that could be formed

(a) if there is no restriction on the letters or digits that can be used;

(b) if no letter or digit can be repeated.

SOLUTION Consider each license plate as having six blanks to be filled in: three letters followed by three numerical digits.

(a) If there are no restrictions on letters or digits, then we can fill in the first blank 26 ways, the second blank 26 ways, the third blank 26 ways, the fourth blank 10 ways, the fifth blank 10 ways, and the sixth blank 10 ways. By the Multiplication Principle, we can fill in all six blanks in $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$ ways. There are 17,576,000 possible license plates with no restrictions on letters or digits.

(b) If no letter or digit can be repeated, then we can fill in the first blank 26 ways, the second blank 25 ways, the third blank 24 ways, the fourth blank 10 ways, the fifth blank 9 ways, and the sixth blank 8 ways. By the Multiplication Principle, we can fill in all six blanks in $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11{,}232{,}000$ ways. There are 11,232,000 possible license plates with no letters or digits repeated.

Now try Exercise 5.

License Plate Restrictions

Although prohibiting repeated letters and digits as in Example 2 would make no practical sense (why rule out more than 6 million possible plates for no good reason?), states do impose some restrictions on license plates. They rule out certain letter progressions that could be considered obscene or offensive.

Permutations

One important application of the Multiplication Principle of Counting is to count the number of ways that a set of n objects (called an ***n*-set**) can be arranged in order. Each such ordering is called a **permutation** of the set. Example 1 showed that there are $3! = 6$ permutations of a 3-set. In fact, if you understand the tree diagram, you can probably guess how many permutations there are of an n-set.

Permutations of an n-set

There are $n!$ permutations of an n-set.

Factorials

If n is a positive integer, the symbol $n!$ (read "n factorial") represents the product $n(n-1)(n-2)(n-3)\cdots 2 \cdot 1$. We also define $0! = 1$.

Usually the elements of a set are distinguishable from one another, but we can adjust our counting when they are not, as we see in Example 3.

EXAMPLE 3 Distinguishable Permutations

Count the number of different 9-letter "words" (don't worry about whether they're in the dictionary) that can be formed using the letters in each word.

(a) DRAGONFLY **(b)** BUTTERFLY **(c)** BUMBLEBEE

SOLUTION

(a) Each permutation of the 9 letters forms a different word. There are $9! = 362{,}880$ such permutations.

(b) There are also 9! permutations of these letters, but a simple permutation of the two T's does not result in a new word. We correct for the overcount by dividing by 2!. There are $\frac{9!}{2!} = 181{,}440$ *distinguishable* permutations of the letters in BUTTERFLY.

(c) Again there are 9! permutations, but the three B's are indistinguishable, as are the three E's, so we divide by 3! twice to correct for the overcount. There are $\frac{9!}{3!3!} = 10{,}080$ distinguishable permutations of the letters in BUMBLEBEE.

Now try Exercise 9.

Distinguishable Permutations

There are $n!$ distinguishable permutations of an n-set containing n distinguishable objects.

If an n-set contains n_1 objects of a first kind, n_2 objects of a second kind, and so on, with $n_1 + n_2 + \cdots + n_k = n$, then the number of distinguishable permutations of the n-set is

$$\frac{n!}{n_1!n_2!n_3!\cdots n_k!}.$$

In many counting problems, we are interested in using n objects to fill r blanks in order, where $r < n$. These are called **permutations of *n* objects taken *r* at a time**. The procedure for counting them is the same; only this time we run out of blanks before we run out of objects.

The first blank can be filled in n ways, the second in $n - 1$ ways, and so on until we come to the rth blank, which can be filled in $n - (r - 1)$ ways. By the Multiplication Principle, we can fill in all r blanks in $n(n - 1)(n - 2)\cdots(n - r + 1)$ ways. This expression can be written in a more compact (but less easily computed) way as $n!/(n - r)!$.

Permutations on a Calculator

Most modern calculators have an ${}_nP_r$ selection built in. They also compute factorials, but remember that factorials get very large. If you want to count the number of permutations of 90 objects taken 5 at a time, be sure to use the ${}_nP_r$ function. The expression 90!/85! is likely to lead to an overflow error.

Permutation Counting Formula

The number of permutations of n objects taken r at a time is denoted ${}_nP_r$ and is given by

$${}_nP_r = \frac{n!}{(n - r)!} \quad \text{for} \quad 0 \le r \le n.$$

If $r > n$, then ${}_nP_r = 0$.

Notice that ${}_nP_n = n!/(n - n)! = n!/0! = n!/1 = n!$, which we have already seen is the number of permutations of a complete set of n objects. This is why we define $0! = 1$.

Notes on Example

Example 4 shows some paper-and-pencil methods for calculating permutations. It is important that students have the algebraic skills to perform these operations, since the numbers in some counting problems may exceed the capacity of a calculator.

EXAMPLE 4 Counting Permutations

Evaluate each expression without a calculator.

(a) ${}_6P_4$ **(b)** ${}_{11}P_3$ **(c)** ${}_nP_3$

SOLUTION

(a) By the formula, ${}_6P_4 = 6!/(6 - 4)! = 6!/2! = (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!)/2! = 6 \cdot 5 \cdot 4 \cdot 3 = 360$.

(b) Although you could use the formula again, you might prefer to apply the Multiplication Principle directly. We have 11 objects and 3 blanks to fill:

$${}_{11}P_3 = 11 \cdot 10 \cdot 9 = 990$$

(c) This time it is definitely easier to use the Multiplication Principle. We have n objects and 3 blanks to fill; so assuming $n \ge 3$,

$${}_nP_3 = n(n - 1)(n - 2).$$

Now try Exercise 15.

EXAMPLE 5 Applying Permutations

Sixteen actors answer a casting call to try out for roles as dwarfs in a production of *Snow White and the Seven Dwarfs*. In how many different ways can the director cast the seven roles?

SOLUTION The 7 different roles can be thought of as 7 blanks to be filled, and we have 16 actors with which to fill them. The director can cast the roles in ${}_{16}P_7 = 57{,}657{,}600$ ways. ***Now try Exercise 12.***

Combinations

When we count permutations of n objects taken r at a time, we consider different orderings of the same r selected objects as being different permutations. In many applications we are only interested in the ways to *select* the r objects, regardless of the order in which we arrange them. These unordered selections are called **combinations of n objects taken r at a time**.

Combination Counting Formula

The number of combinations of n objects taken r at a time is denoted ${}_nC_r$ and is given by

$${}_nC_r = \frac{n!}{r!(n-r)!} \quad \text{for} \quad 0 \le r \le n.$$

If $r > n$, then ${}_nC_r = 0$.

We can verify the ${}_nC_r$ formula with the Multiplication Principle. Since every permutation can be thought of as an *unordered* selection of r objects *followed* by a particular *ordering* of the objects selected, the Multiplication Principle gives ${}_nP_r = {}_nC_r \cdot r!$.

Therefore

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{1}{r!} \cdot \frac{n!}{(n-r)!} = \frac{n!}{r!(n-r)!}.$$

A Word on Notation

Some textbooks use $P(n, r)$ instead of ${}_nP_r$ and $C(n, r)$ instead of ${}_nC_r$. Much more common is the notation $\binom{n}{r}$ for ${}_nC_r$. Both $\binom{n}{r}$ and ${}_nC_r$ are often read "n choose r."

Combinations on a Calculator

Most modern calculators have an nCr selection built in. As with permutations, it is better to use the nCr function than to use the formula $\frac{n!}{r!(n-r)!}$, as the individual factorials can get too large for the calculator.

EXAMPLE 6 Distinguishing Combinations from Permutations

In each of the following scenarios, tell whether permutations (ordered) or combinations (unordered) are being described.

(a) A president, vice-president, and secretary are chosen from a 25-member garden club.

(b) A cook chooses 5 potatoes from a bag of 12 potatoes to make a potato salad.

(c) A teacher makes a seating chart for 22 students in a classroom with 30 desks.

SOLUTION

(a) Permutations. Order matters because it matters who gets which office.

(b) Combinations. The salad is the same no matter what order the potatoes are chosen.

(c) Permutations. A different ordering of students in the same seats results in a different seating chart.

Notice that once you know what is being counted, getting the correct number is easy with a calculator. The number of possible choices in the scenarios above are: (a) ${}_{25}P_3 = 13{,}800$, (b) ${}_{12}C_5 = 792$, and (c) ${}_{30}P_{22} \approx 6.5787 \times 10^{27}$.

Now try Exercise 19.

EXAMPLE 7 Counting Combinations

In the Miss America pageant, 51 contestants must be narrowed down to 10 finalists who will compete on national television. In how many possible ways can the 10 finalists be selected?

SOLUTION Notice that the *order* of the finalists does not matter at this phase; all that matters is which women are selected. So we count combinations rather than permutations.

$$_{51}C_{10} = \frac{51!}{10!41!} = 12{,}777{,}711{,}870$$

The 10 finalists can be chosen in 12,777,711,870 ways. *Now try Exercise 27.*

EXAMPLE 8 Picking Lottery Numbers

The Georgia Lotto requires winners to pick 6 integers between 1 and 46. The order in which you select them does not matter; indeed, the lottery tickets are always printed with the numbers in ascending order. How many different lottery tickets are possible?

SOLUTION There are $_{46}C_6 = 9{,}366{,}819$ possible lottery tickets of this type. (That's more than enough different tickets for every person in the state of Georgia!) *Now try Exercise 29.*

Subsets of an *n*-Set

As a final application of the counting principle, consider the pizza topping problem.

EXAMPLE 9 Selecting Pizza Toppings

Armando's Pizzeria offers patrons any combination of up to 10 different toppings: pepperoni, mushroom, sausage, onion, green pepper, bacon, prosciutto, black olive, green olive, and anchovies. How many different pizzas can be ordered

(a) if we can choose any three toppings?

(b) if we can choose any number of toppings (0 through 10)?

SOLUTION

(a) Order does not matter (for example, the sausage-pepperoni-mushroom pizza is the same as the pepperoni-mushroom-sausage pizza), so the number of possible pizzas is $_{10}C_3 = 120$.

(b) We could add up all the numbers of the form $_{10}C_r$ for $r = 0, 1, \dots, 10$, but there is an easier way to count the possibilities. Consider the ten options to be lined up as in the statement of the problem. In considering each option, we have two choices: yes or no. (For example, the pepperoni-mushroom-sausage pizza would correspond to the sequence YYYNNNNNNN.) By the Multiplication Principle, the number of such sequences is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 1024$, which is the number of possible pizzas. *Now try Exercise 37.*

The solution to Example 9b suggests a general rule that will be our last counting formula of the section.

Formula for Counting Subsets of an *n*-Set

There are 2^n subsets of a set with n objects (including the empty set and the entire set).

EXAMPLE 10 Analyzing an Advertised Claim

A national hamburger chain used to advertise that it fixed its hamburgers "256 ways," since patrons could order whatever toppings they wanted. How many toppings must have been available?

SOLUTION We need to solve the equation $2^n = 256$ for n. We could solve this easily enough by trial and error, but we will solve it with logarithms just to keep the method fresh in our minds.

$$\begin{aligned} 2^n &= 256 \\ \log 2^n &= \log 256 \\ n \log 2 &= \log 256 \\ n &= \frac{\log 256}{\log 2} \\ n &= 8 \end{aligned}$$

There must have been 8 toppings from which to choose.

Now try Exercise 39.

Why Are There Not 1000 Possible Area Codes?

While there are 1000 three-digit numbers between 000 and 999, not all of them are available for use as area codes. For example, area codes cannot begin with 0 or 1, and numbers of the form *abb* have been reserved for other purposes.

Chapter Opener Problem (from page 641)

Problem: There are 680 three-digit numbers that are available for use as area codes in North America. As of April 2002, 305 of them were actually being used (*Source: www.nanpa.com*). How many additional three-digit area codes are available for use? Within a given area code, how many unique telephone numbers are theoretically possible?

Solution: There are $680 - 305 = 375$ additional area codes available. Within a given area code, each telephone number has seven digits chosen from the ten digits 0 through 9. Since each digit can theoretically be any of 10 numbers, there are

$$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^7 = 10{,}000{,}000$$

different telephone numbers possible within a given area code.

Putting these two results together, we see that the unused area codes in April 2002 represented an additional 3.75 billion possible telephone numbers!

QUICK REVIEW 9.1

In Exercises 1–10, give the number of objects described. In some cases you might have to do a little research or ask a friend.

1. The number of cards in a standard deck
2. The number of cards of each suit in a standard deck
3. The number of faces on a cubical die
4. The number of possible totals when two dice are rolled
5. The number of vertices of a decagon
6. The number of musicians in a string quartet
7. The number of players on a soccer team
8. The number of prime numbers between 1 and 10, inclusive
9. The number of squares on a chessboard
10. The number of cards in a contract bridge hand

SECTION 9.1 EXERCISES

In Exercises 1–4, count the number of ways that each procedure can be done.

1. Line up three people for a photograph.
2. Prioritize four pending jobs from most to least important.
3. Arrange five books from left to right on a bookshelf.
4. Award ribbons for 1st place to 5th place to the top five dogs in a dog show.
5. **Homecoming King and Queen** There are four candidates for homecoming queen and three candidates for king. How many king-queen pairs are possible?
6. **Possible Routes** There are three roads from town *A* to town *B* and four roads from town *B* to town *C*. How many different routes are there from *A* to *C* by way of *B*?
7. **Permuting Letters** How many 9-letter "words" (not necessarily in any dictionary) can be formed from the letters of the word LOGARITHM? (Curiously, one such arrangement spells another word related to mathematics. Can you name it?)
8. **Three-Letter Crossword Entries** Excluding J, Q, X, and Z, how many 3-letter crossword puzzle entries can be formed that contain no repeated letters? (It has been conjectured that all of them have appeared in puzzles over the years, sometimes with painfully contrived definitions.)
9. **Permuting Letters** How many distinguishable 11-letter "words" can be formed using the letters in MISSISSIPPI?
10. **Permuting Letters** How many distinguishable 11-letter "words" can be formed using the letters in CHATTANOOGA?
11. **Electing Officers** The 13 members of the East Brainerd Garden Club are electing a President, Vice-President, and Secretary from among their members. How many different ways can this be done?
12. **City Government** From among 12 projects under consideration, the mayor must put together a prioritized (that is, ordered) list of 6 projects to submit to the city council for funding. How many such lists can be formed?

In Exercises 13–18, evaluate each expression without a calculator. Then check with your calculator to see if your answer is correct.

13. $4!$
14. $(3!)(0!)$
15. ${}_6P_2$
16. ${}_9P_2$
17. ${}_{10}C_7$
18. ${}_{10}C_3$

In Exercises 19–22, tell whether permutations (ordered) or combinations (unordered) are being described.

19. 13 cards are selected from a deck of 52 to form a bridge hand.
20. 7 digits are selected (without repetition) to form a telephone number.
21. 4 students are selected from the senior class to form a committee to advise the cafeteria director about food.
22. 4 actors are chosen to play the Beatles in a film biography.
23. **License Plates** How many different license plates begin with two digits, followed by two letters and then three digits if no letters or digits are repeated?
24. **License Plates** How many different license plates consist of five symbols, either digits or letters?
25. **Tumbling Dice** Suppose that two dice, one red and one green, are rolled. How many different outcomes are possible for the pair of dice?
26. **Coin Toss** How many different sequences of heads and tails are there if a coin is tossed 10 times?
27. **Forming Committees** A 3-woman committee is to be elected from a 25-member sorority. How many different committees can be elected?
28. **Straight Poker** In the original version of poker known as "straight" poker, a five-card hand is dealt from a standard deck of 52. How many different straight poker hands are possible?
29. **Buying Discs** Juan has money to buy only three of the 48 compact discs available. How many different sets of discs can he purchase?
30. **Coin Toss** A coin is tossed 20 times and the heads and tails sequence is recorded. From among all the possible sequences of heads and tails, how many have exactly seven heads?
31. **Drawing Cards** How many different 13-card hands include the ace and king of spades?
32. **Job Interviews** The head of the personnel department interviews eight people for three identical openings. How many different groups of three can be employed?
33. **Scholarship Nominations** Six seniors at Rydell High School meet the qualifications for a competitive honor scholarship at a major university. The university allows the school to nominate up to three candidates, and the school always nominates at least one. How many different choices could the nominating committee make?
34. **Pu-pu Platters** A Chinese restaurant will make a Pu-pu platter "to order" containing any one, two, or three selections from its appetizer menu. If the menu offers five different appetizers, how many different platters could be made?

35. **Yahtzee** In the game of Yahtzee, five dice are tossed simultaneously. How many outcomes can be distinguished if all the dice are different colors?
36. **Indiana Jones and the Final Exam** Professor Indiana Jones gives his class 20 study questions, from which he will select 8 to be answered on the final exam. How many ways can he select the questions?

37. **Salad Bar** Mary's lunch always consists of a full plate of salad from Ernestine's salad bar. She always takes equal amounts of each salad she chooses, but she likes to vary her selections. If she can choose from among 9 different salads, how many essentially different lunches can she create?

38. **Buying a New Car** A new car customer has to choose from among 3 models, each of which comes in 4 exterior colors, 3 interior colors, and with any combination of up to 6 optional accessories. How many essentially different ways can the customer order the car?

39. **Pizza Possibilities** Luigi sells one size of pizza, but he claims that his selection of toppings allows for "more than 4000 different choices." What is the smallest number of toppings Luigi could offer?

40. **Proper Subsets** A subset of set A is called *proper* if it is neither the empty set nor the entire set A. How many proper subsets does an n-set have?

41. **True-False Tests** How many different answer keys are possible for a 10-question true-false test?

42. **Multiple-Choice Tests** How many different answer keys are possible for a 10-question multiple-choice test in which each question leads to choice a, b, c, d, or e?

Standardized Test Questions

43. **True or False** If a and b are positive integers such that $a + b = n$, then $\binom{n}{a} = \binom{n}{b}$. Justify your answer.

44. **True or False** If a, b, and n are integers such that $a < b < n$, then $\binom{n}{a} < \binom{n}{b}$. Justify your answer.

You may use a graphing calculator when evaluating Exercises 45–48.

45. **Multiple Choice** Lunch at the Gritsy Palace consists of an entrée, two vegetables, and a dessert. If there are four entrées, six vegetables, and six desserts from which to choose, how many essentially different lunches are possible?

(A) 16

(B) 25

(C) 144

(D) 360

(E) 720

46. **Multiple Choice** How many different ways can the judges choose 5th to 1st places from ten Miss America finalists?

(A) 50

(B) 120

(C) 252

(D) 30,240

(E) 3,628,800

47. **Multiple Choice** Assuming r and n are positive integers with $r < n$, which of the following numbers does *not* equal 1?

(A) $(n - n)!$

(B) ${}_nP_n$

(C) ${}_nC_n$

(D) $\binom{n}{n}$

(E) $\binom{n}{r} \div \binom{n}{n-r}$

48. **Multiple Choice** An organization is electing 3 new board members by approval voting. Members are given ballots with the names of 5 candidates and are allowed to check off the names of all candidates whom they would approve (which could be none, or even all five). The three candidates with the most checks overall are elected. In how many different ways can a member fill out the ballot?

(A) 10

(B) 20

(C) 32

(D) 125

(E) 243

Explorations

49. **Group Activity** For each of the following numbers, make up a counting problem that has the number as its answer.

(a) ${}_{52}C_3$

(b) ${}_{12}C_3$

(c) ${}_{25}P_{11}$

(d) 2^5

(e) $3 \cdot 2^{10}$

50. **Writing to Learn** You have a fresh carton containing one dozen eggs and you need to choose two for breakfast. Give a counting argument based on this scenario to explain why ${}_{12}C_2 = {}_{12}C_{10}$.

51. **Factorial Riddle** The number 50! ends in a string of consecutive 0's.

(a) How many 0's are in the string?

(b) How do you know?

52. **Group Activity Diagonals of a Regular Polygon** In Exploration 1 of Section 1.7, you reasoned from data points and quadratic regression that the number of diagonals of a regular polygon with n vertices was $(n^2 - 3n)/2$.

(a) Explain why the number of segments connecting all pairs of vertices is ${}_nC_2$.

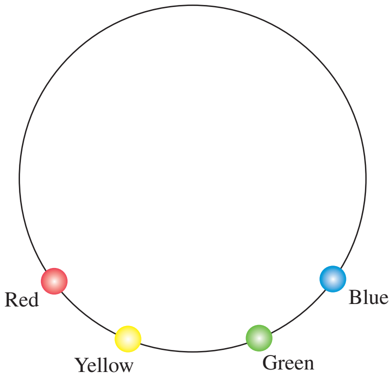
(b) Use the result from part (a) to prove that the number of diagonals is $(n^2 - 3n)/2$.

Extending the Ideas

53. Writing to Learn Suppose that a chain letter (illegal if money is involved) is sent to five people the first week of the year. Each of these five people sends a copy of the letter to five more people during the second week of the year. Assume that everyone who receives a letter participates. Explain how you know with certainty that someone will receive a second copy of this letter later in the year.

54. A Round Table How many different seating arrangements are possible for 4 people sitting around a round table?

55. Colored Beads Four beads—red, blue, yellow, and green—are arranged on a string to make a simple necklace as shown in the figure. How many arrangements are possible?



56. Casting a Play A director is casting a play with two female leads and wants to have a chance to audition the actresses two at a time to get a feeling for how well they would work together. His casting director and his administrative assistant both prepare charts to show the amount of time that would be required, depending on the number of actresses who come to the audition. Which time chart is more reasonable, and why?

Number Who Audition	Time Required (minutes)
3	10
6	45
9	110
12	200
15	320

Number Who Audition	Time Required (minutes)
3	10
6	30
9	60
12	100
15	150

57. Bridge Around the World Suppose that a contract bridge hand is dealt somewhere in the world every second. What is the fewest number of years required for every possible bridge hand to be dealt? (See Quick Review Exercise 10.)

58. Basketball Lineups Each NBA basketball team has 12 players on its roster. If each coach chooses 5 starters without regard to position, how many different sets of 10 players can start when two given teams play a game?