8.3 Hyperbolas

What you'll learn about

- Geometry of a Hyperbola
- Translations of Hyperbolas
- Eccentricity and Orbits
- Reflective Property of a Hyperbola
- Long-Range Navigation

. . . and why

The hyperbola is the least-known conic section, yet it is used in astronomy, optics, and navigation.

Geometry of a Hyperbola

When a plane intersects both nappes of a right circular cylinder, the intersection is a hyperbola. The definition, features, and derivation for a hyperbola closely resemble those for an ellipse. As you read on, you may find it helpful to compare the nature of the hyperbola with the nature of the ellipse.

DEFINITION Hyperbola

A hyperbola is the set of all points in a plane whose distances from two fixed points in the plane have a constant *difference*. The fixed points are the **foci** of the hyperbola. The line through the foci is the **focal axis**. The point on the focal axis midway between the foci is the **center**. The points where the hyperbola intersects its focal axis are the **vertices** of the hyperbola (Figure 8.22).

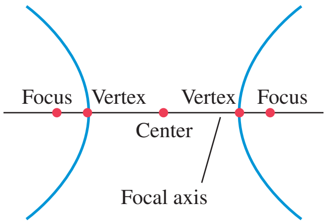


FIGURE 8.22 Key points on the focal axis of a hyperbola.

Figure 8.23 shows a hyperbola centered at the origin with its focal axis on the x-axis. The vertices are at $(-a, 0)$ and $(a, 0)$, where a is some positive constant. The fixed points $F_1(-c, 0)$ and $F_2(c, 0)$ are the foci of the hyperbola, with $c > a$.

Notice that the hyperbola has two *branches*. For a point $P(x, y)$ on the right-hand branch, $PF_1 - PF_2 = 2a$. On the left-hand branch, $PF_2 - PF_1 = 2a$. Combining these two equations gives us

$$PF_1 - PF_2 = \pm 2a.$$

Using the distance formula, the equation becomes

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = \pm 2a.$$

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2} \qquad \text{Rearrange terms.}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \qquad \text{Square.}$$

$$\mp a\sqrt{(x+c)^2 + y^2} = a^2 + cx \qquad \text{Simplify.}$$

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2 \qquad \text{Square.}$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \qquad \text{Multiply by } -1 \text{ and simplify.}$$

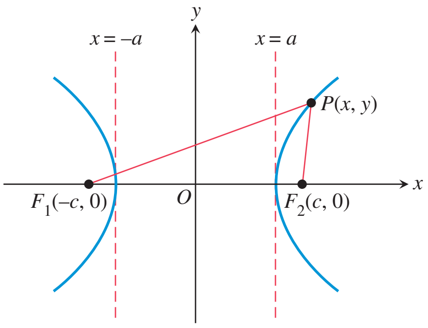


FIGURE 8.23 *Structure of a Hyperbola.* The difference of the distances from the foci to each point on the hyperbola is a constant.

Letting $b^2 = c^2 - a^2$, we have

$$b^2x^2 - a^2y^2 = a^2b^2,$$

which is usually written as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Because these steps can be reversed, a point $P(x, y)$ satisfies this last equation if and only if the point lies on the hyperbola defined by $PF_1 - PF_2 = \pm 2a$, provided that

$c > a > 0$ and $b^2 = c^2 - a^2$. The *Pythagorean relation* $b^2 = c^2 - a^2$ can be written many ways, including $a^2 = c^2 - b^2$ and $c^2 = a^2 + b^2$.

The equation $x^2/a^2 - y^2/b^2 = 1$ is the **standard form** of the equation of a hyperbola centered at the origin with the x-axis as its focal axis. A hyperbola centered at the origin with the y-axis as its focal axis is the *inverse relation* of $x^2/a^2 - y^2/b^2 = 1$, and thus has an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

As with other conics, a line segment with endpoints on a hyperbola is a **chord** of the hyperbola. The chord lying on the focal axis connecting the vertices is the **transverse axis** of the hyperbola. The length of the transverse axis is $2a$. The line segment of length $2b$ that is perpendicular to the focal axis and that has the center of the hyperbola as its midpoint is the **conjugate axis** of the hyperbola. The number a is the **semitransverse axis**, and b is the **semiconjugate axis**.

Naming Axes

The word "transverse" comes from the Latin *trans vertere*: to go across. The transverse axis "goes across" from one vertex to the other. The conjugate axis is the transverse axis for the *conjugate hyperbola*, defined in Exercise 73.

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two *asymptotes*. These asymptotes are slant lines that can be found by replacing the 1 on the right-hand side of the hyperbola's equation by a 0:

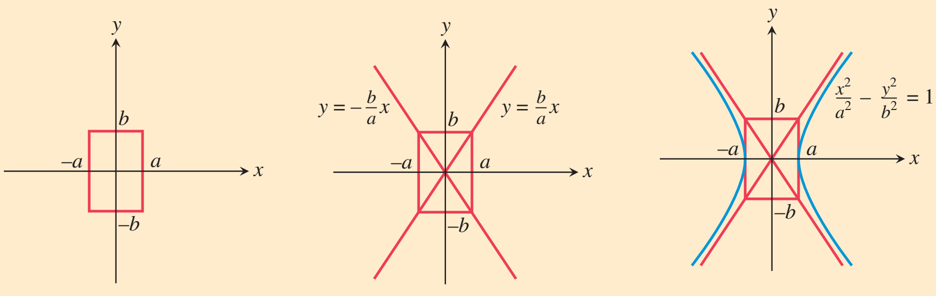
$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}} \rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{Replace 1 by 0.}} \rightarrow \underbrace{y = \pm\frac{b}{a}x}_{\text{asymptotes}}$$

A hyperbola centered at the origin with its focal axis one of the coordinate axes is symmetric with respect to the origin and both coordinate axes. Such a hyperbola can be sketched by drawing a rectangle centered at the origin with sides parallel to the coordinate axes, followed by drawing the asymptotes through opposite corners of the rectangle, and finally sketching the hyperbola using the central rectangle and asymptotes as guides, as shown in the Drawing Lesson.

Drawing Lesson

How to Sketch the Hyperbola $x^2/a^2 - y^2/b^2 = 1$

1. Sketch line segments at $x = \pm a$ and $y = \pm b$, and complete the rectangle they determine.
2. Sketch the asymptotes by extending the rectangle's diagonals.
3. Use the rectangle and asymptotes to guide your drawing.



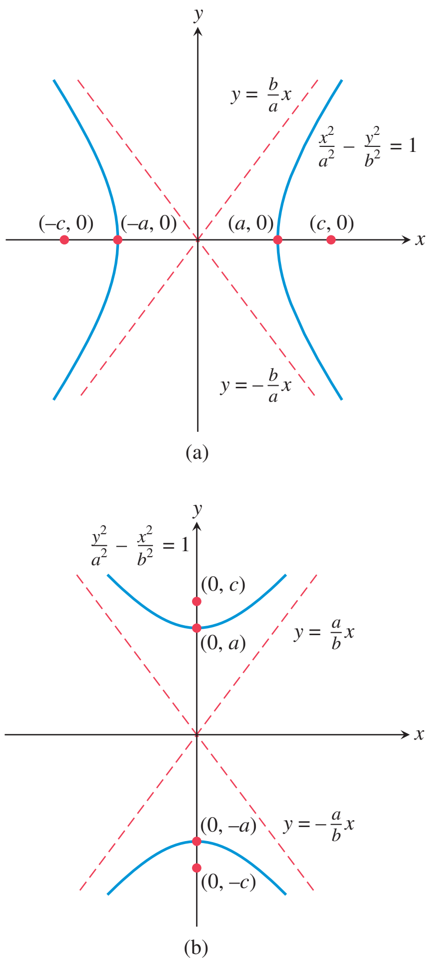


FIGURE 8.24 Hyperbolas centered at the origin with foci on (a) the x-axis and (b) the y-axis.

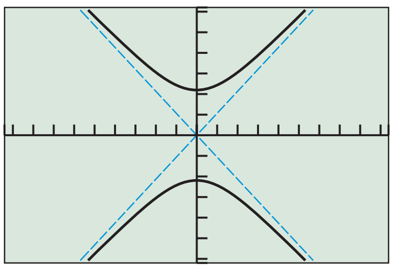


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FIGURE 8.25 The hyperbola $y^2/5 - x^2/4 = 1$, shown with its asymptotes. (Example 2)

Hyperbolas with Center (0, 0)

• **Standard equation**	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
• **Focal axis**	x-axis	y-axis
• **Foci**	$(\pm c, 0)$	$(0, \pm c)$
• **Vertices**	$(\pm a, 0)$	$(0, \pm a)$
• **Semitransverse axis**	a	a
• **Semiconjugate axis**	b	b
• **Pythagorean relation**	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
• **Asymptotes**	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

See Figure 8.24.

EXAMPLE 1 Finding the Vertices and Foci of a Hyperbola

Find the vertices and the foci of the hyperbola $4x^2 - 9y^2 = 36$.

SOLUTION Dividing both sides of the equation by 36 yields the standard form $x^2/9 - y^2/4 = 1$. So $a^2 = 9$, $b^2 = 4$, and $c^2 = a^2 + b^2 = 9 + 4 = 13$. Thus the vertices are $(\pm 3, 0)$, and the foci are $(\pm\sqrt{13}, 0)$. ***Now try Exercise 1.***

If we wish to graph a hyperbola using a function grapher, we need to solve the equation of the hyperbola for y, as illustrated in Example 2.

EXAMPLE 2 Finding an Equation and Graphing a Hyperbola

Find an equation of the hyperbola with foci $(0, -3)$ and $(0, 3)$ whose conjugate axis has length 4. Sketch the hyperbola and its asymptotes, and support your sketch with a grapher.

SOLUTION The center is $(0, 0)$. The foci are on the y-axis with $c = 3$. The semiconjugate axis is $b = 4/2 = 2$. Thus $a^2 = c^2 - b^2 = 3^2 - 2^2 = 5$. The standard form of the equation for the hyperbola is

$$\frac{y^2}{5} - \frac{x^2}{4} = 1.$$

Using $a = \sqrt{5} \approx 2.24$ and $b = 2$, we can sketch the central rectangle, the asymptotes, and the hyperbola itself. Try doing this. To graph the hyperbola using a function grapher, we solve for y in terms of x.

$$\frac{y^2}{5} = 1 + \frac{x^2}{4} \qquad \text{Add } \frac{x^2}{4}$$

$$y^2 = 5(1 + x^2/4) \qquad \text{Multiply by 5.}$$

$$y = \pm\sqrt{5(1 + x^2/4)} \qquad \text{Extract square roots.}$$

Figure 8.25 shows the graphs of

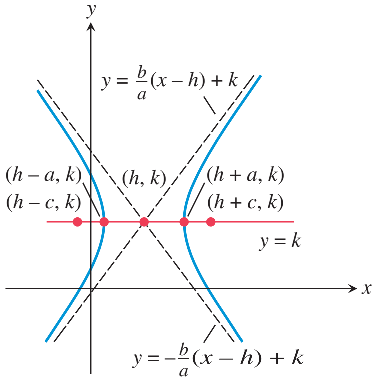
$$y_1 = \sqrt{5(1 + x^2/4)} \quad \text{and} \quad y_2 = -\sqrt{5(1 + x^2/4)},$$

together with the asymptotes of the hyperbola

$$y_3 = \frac{\sqrt{5}}{2}x \quad \text{and} \quad y_4 = -\frac{\sqrt{5}}{2}x.$$

Now try Exercise 17.

In Example 2, because the hyperbola had a vertical focal axis, selecting a viewing rectangle was easy. When a hyperbola has a horizontal focal axis, we try to select a viewing window to include the two vertices in the plot and thus avoid gaps in the graph of the hyperbola.



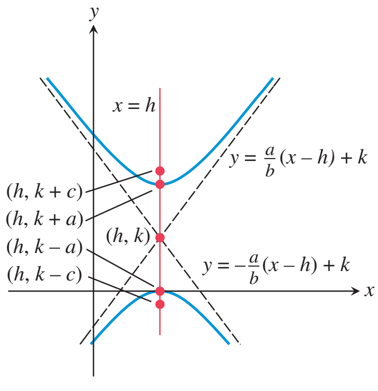


FIGURE 8.26 Hyperbolas with center (h, k) and foci on (a) $y = k$ and (b) $x = h$.

Translations of Hyperbolas

When a hyperbola with center $(0, 0)$ is translated horizontally by h units and vertically by k units, the center of the hyperbola moves from $(0, 0)$ to (h, k), as shown in Figure 8.26. Such a translation does not change the length of the transverse or conjugate axis or the Pythagorean relation.

Hyperbolas with Center (h, k)

• **Standard equation**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
• **Focal axis**	$y = k$	$x = h$
• **Foci**	$(h \pm c, k)$	$(h, k \pm c)$
• **Vertices**	$(h \pm a, k)$	$(h, k \pm a)$
• **Semitransverse axis**	a	a
• **Semiconjugate axis**	b	b
• **Pythagorean relation**	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
• **Asymptotes**	$y = \pm\frac{b}{a}(x - h) + k$	$y = \pm\frac{a}{b}(x - h) + k$

See Figure 8.26.

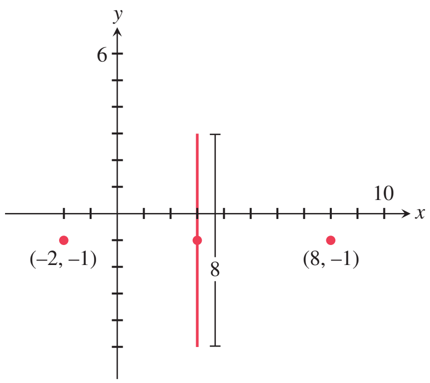


FIGURE 8.27 Given information for Example 3.

EXAMPLE 3 Finding an Equation of a Hyperbola

Find the standard form of the equation for the hyperbola whose transverse axis has endpoints $(-2, -1)$ and $(8, -1)$ and whose conjugate axis has length 8.

SOLUTION Figure 8.27 shows the transverse axis endpoints, the conjugate axis, and the center of the hyperbola. The standard equation of this hyperbola has the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

where the center (h, k) is the midpoint $(3, -1)$ of the transverse axis. The semitransverse axis and semiconjugate axis are

$$a = \frac{8 - (-2)}{2} = 5 \quad \text{and} \quad b = \frac{8}{2} = 4.$$

So the equation we seek is

$$\frac{(x-3)^2}{5^2} - \frac{(y-(-1))^2}{4^2} = 1,$$

$$\frac{(x-3)^2}{25} - \frac{(y+1)^2}{16} = 1.$$

Now try Exercise 31.

EXAMPLE 4 Locating Key Points of a Hyperbola

Find the center, vertices, and foci of the hyperbola

$$\frac{(x+2)^2}{9} - \frac{(y-5)^2}{49} = 1.$$

SOLUTION The center (h, k) is $(-2, 5)$. Because the semitransverse axis $a = \sqrt{9} = 3$, the vertices are

$$(h + a, k) = (-2 + 3, 5) = (1, 5) \quad \text{and}$$
$$(h - a, k) = (-2 - 3, 5) = (-5, 5).$$

Because $c = \sqrt{a^2 + b^2} = \sqrt{9 + 49} = \sqrt{58}$, the foci $(h \pm c, k)$ are $(-2 \pm \sqrt{58}, 5)$, or approximately $(5.62, 5)$ and $(-9.62, 5)$. *Now try Exercise 39.*

With the information found about the hyperbola in Example 4 and knowing that its semiconjugate axis $b = \sqrt{49} = 7$, we could easily sketch the hyperbola. Obtaining an accurate graph of the hyperbola using a function grapher is another matter. Often, the best way to graph a hyperbola using a graphing utility is to use parametric equations.

EXPLORATION 1 Graphing a Hyperbola Using Its Parametric Equations

1. Use the Pythagorean trigonometry identity $\sec^2 t - \tan^2 t = 1$ to prove that the parametrization $x = -1 + 3/\cos t$, $y = 1 + 2 \tan t$ $(0 \le t \le 2\pi)$ will produce a graph of the hyperbola $(x + 1)^2/9 - (y - 1)^2/4 = 1$.
2. Using Dot graphing mode, graph $x = -1 + 3/\cos t$, $y = 1 + 2 \tan t$ $(0 \le t \le 2\pi)$ in a square viewing window to support part 1 graphically. Switch to Connected graphing mode, and regraph the equation. What do you observe? Explain.
3. Create parametrizations for the hyperbolas in Examples 1, 2, 3, and 4.
4. Graph each of your parametrizations in part 3 and check the features of the obtained graph to see whether they match the expected geometric features of the hyperbola. If necessary, revise your parametrization and regraph until all features match.
5. Prove that each of your parametrizations is valid.

Eccentricity and Orbits

DEFINITION Eccentricity of a Hyperbola

The **eccentricity** of a hyperbola is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a},$$

where a is the semitransverse axis, b is the semiconjugate axis, and c is the distance from the center to either focus.

For a hyperbola, because $c > a$, the eccentricity $e > 1$. In Section 8.2 we learned that the eccentricity of an ellipse satisfies the inequality $0 \le e < 1$ and that, for $e = 0$, the ellipse is a circle. In Section 8.5 we will generalize the concept of eccentricity to all types of conics and learn that the eccentricity of a parabola is $e = 1$.

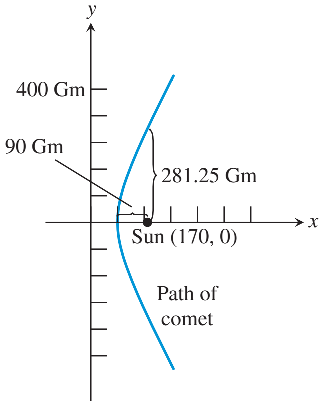


FIGURE 8.28 The graph of one branch of $x^2/6400 - y^2/22{,}500 = 1$. (Example 5)

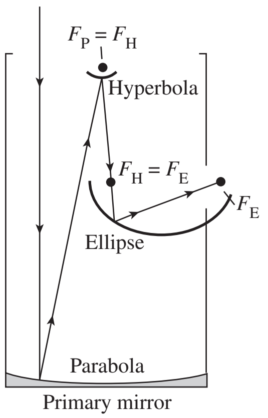


FIGURE 8.29 Cross section of a reflecting telescope.

Kepler's First Law of Planetary Motion says that a planet's orbit is elliptical with the Sun at one focus. Since 1609, astronomers have generalized Kepler's Law; the current theory states: A celestial body that travels within the gravitational field of a much more massive body follows a path that closely approximates a conic section that has the more massive body as a focus. Two bodies that do not differ greatly in mass (such as Earth and the Moon, or Pluto and its moon Charon) actually revolve around their balance point, or *barycenter*. In theory, a comet can approach the Sun from interstellar space, make a partial loop about the Sun, and then leave the solar system, returning to deep space; such a comet follows a path that is one branch of a hyperbola.

EXAMPLE 5 Analyzing a Comet's Orbit

A comet following a hyperbolic path about the Sun has a perihelion distance of 90 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 281.25 Gm from the Sun. Calculate a, b, c, and e. What are the coordinates of the center of the Sun if we coordinatize space so that the hyperbola is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1?$$

SOLUTION The perihelion distance is $c - a = 90$. When $x = c$, $y = \pm b^2/a$ (see Exercise 74). So $b^2/a = 281.25$, or $b^2 = 281.25a$. Because $b^2 = c^2 - a^2$, we have the system

$$c - a = 90 \quad \text{and} \quad c^2 - a^2 = 281.25a,$$

which yields the equation:

$$\begin{aligned} (a + 90)^2 - a^2 &= 281.25a \\ a^2 + 180a + 8100 - a^2 &= 281.25a \\ 8100 &= 101.25a \\ a &= 80 \end{aligned}$$

Therefore, $a = 80$ Gm, $b = 150$ Gm, $c = 170$ Gm, and $e = 17/8 = 2.125$ (Figure 8.28). If the comet's path is the branch of the hyperbola with positive x-coordinates, then the Sun is at the focus $(c, 0) = (170, 0)$. *Now try Exercise 55.*

Reflective Property of a Hyperbola

Like other conics, a hyperbola can be used to make a reflector of sound, light, and other waves. If we rotate a hyperbola in three-dimensional space about its focal axis, the hyperbola sweeps out a **hyperboloid of revolution**. If a signal is directed toward a focus of a reflective hyperboloid, the signal reflects off the hyperbolic surface to the other focus. In Figure 8.29 light reflects off a primary parabolic mirror toward the mirror's focus $F_P = F_H$, which is also the focus of a small hyperbolic mirror. The light is then reflected off the hyperbolic mirror, toward the hyperboloid's other focus $F_H = F_E$, which is also the focus of an elliptical mirror. Finally the light is reflected into the observer's eye, which is at the second focus of the ellipsoid F_E.

Reflecting telescopes date back to the 1600s when Isaac Newton used a primary parabolic mirror in combination with a flat secondary mirror, slanted to reflect the light out the side to the eyepiece. French optician G. Cassegrain was the first to use a hyperbolic secondary mirror, which directed the light through a hole at the vertex of the primary mirror (see Exercise 70). Today, reflecting telescopes such as the Hubble Space Telescope have become quite sophisticated and must have nearly perfect mirrors to focus properly.

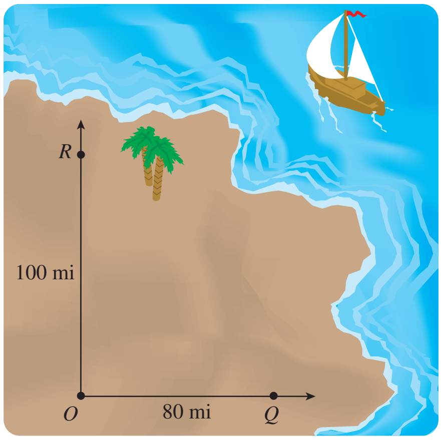


FIGURE 8.30 Strategically located LORAN transmitters O, Q, and R. (Example 6)

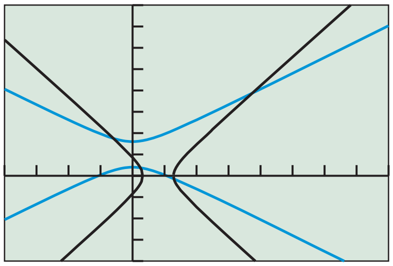


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FIGURE 8.31 Graphs for Example 6.

Long-Range Navigation

Hyperbolas and radio signals are the basis of the LORAN (long-range navigation) system. Example 6 illustrates this system using the definition of hyperbola and the fact that radio signals travel 980 ft per microsecond (1 microsecond $= 1\ \mu\text{sec} = 10^{-6}$ sec).

EXAMPLE 6 Using the LORAN System

Radio signals are sent simultaneously from transmitters located at points O, Q, and R (Figure 8.30). R is 100 mi due north of O, and Q is 80 mi due east of O. The LORAN receiver on sloop *Gloria* receives the signal from O 323.27 μsec after the signal from R, and 258.61 μsec after the signal from Q. What is the sloop's bearing and distance from O?

SOLUTION The *Gloria* is at a point of intersection between two hyperbolas: one with foci O and R, the other with foci O and Q.

The hyperbola with foci $O(0, 0)$ and $R(0, 100)$ has center $(0, 50)$ and transverse axis

$$2a = (323.27\ \mu\text{sec})(980\ \text{ft}/\mu\text{sec})(1\ \text{mi}/5280\ \text{ft}) \approx 60\ \text{mi}.$$

Thus $a \approx 30$ and $b = \sqrt{c^2 - a^2} \approx \sqrt{50^2 - 30^2} = 40$, yielding the equation

$$\frac{(y - 50)^2}{30^2} - \frac{x^2}{40^2} = 1.$$

The hyperbola with foci $O(0, 0)$ and $Q(80, 0)$ has center $(40, 0)$ and transverse axis

$$2a = (258.61\ \mu\text{sec})(980\ \text{ft}/\mu\text{sec})(1\ \text{mi}/5280\ \text{ft}) \approx 48\ \text{mi}.$$

Thus $a \approx 24$ and $b = \sqrt{c^2 - a^2} \approx \sqrt{40^2 - 24^2} = 32$, yielding the equation

$$\frac{(x - 40)^2}{24^2} - \frac{y^2}{32^2} = 1.$$

The *Gloria* is at point P where upper and right-hand branches of the hyperbolas meet (Figure 8.31). Using a grapher we find that $P \approx (187.09, 193.49)$. The bearing from point O is

$$\theta \approx 90° - \tan^{-1}\left(\frac{193.49}{187.09}\right) \approx 44.04°,$$

and the distance from point O is

$$d \approx \sqrt{187.09^2 + 193.49^2} \approx 269.15.$$

So the *Gloria* is about 187.1 mi east and 193.5 mi north of point O on a bearing of roughly 44°, and the sloop is about 269 mi from point O. ***Now try Exercise 57.***

QUICK REVIEW 8.3 *(For help, go to Sections P.2, P.5, and 7.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1 and 2, find the distance between the given points.

1. $(4, -3)$ and $(-7, -8)$

2. $(a, -3)$ and (b, c)

In Exercises 3 and 4, solve for y in terms of x.

3. $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

4. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

In Exercises 5–8, solve for x.

5. $\sqrt{3x + 12} - \sqrt{3x - 8} = 10$

6. $\sqrt{4x + 12} - \sqrt{x + 8} = 1$

7. $\sqrt{6x^2 + 12} - \sqrt{6x^2 + 1} = 1$

8. $\sqrt{2x^2 + 12} - \sqrt{3x^2 + 4} = -8$

In Exercises 9 and 10, solve the system of equations.

9. $c - a = 2$ and $c^2 - a^2 = 16a/3$

10. $c - a = 1$ and $c^2 - a^2 = 25a/12$

SECTION 8.3 EXERCISES

In Exercises 1–6, find the vertices and foci of the hyperbola.

1. $\dfrac{x^2}{16} - \dfrac{y^2}{7} = 1$

2. $\dfrac{y^2}{25} - \dfrac{x^2}{21} = 1$

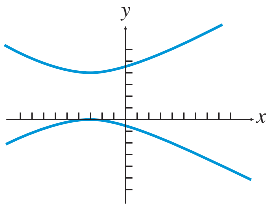
3. $\dfrac{y^2}{36} - \dfrac{x^2}{13} = 1$

4. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

5. $3x^2 - 4y^2 = 12$

6. $9x^2 - 4y^2 = 36$

In Exercises 7–10, match the graph with its equation.



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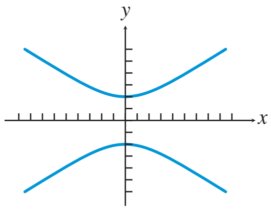

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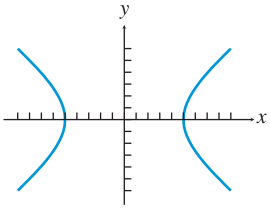

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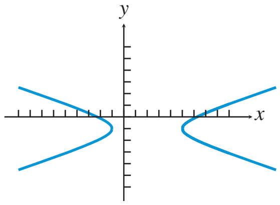


(d)

7. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$

8. $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$

9. $\dfrac{(y-2)^2}{4} - \dfrac{(x+3)^2}{16} = 1$

10. $\dfrac{(x-2)^2}{9} - (y+1)^2 = 1$

In Exercises 11–16, sketch the graph of the hyperbola by hand.

11. $\dfrac{x^2}{49} - \dfrac{y^2}{25} = 1$

12. $\dfrac{y^2}{64} - \dfrac{x^2}{25} = 1$

13. $\dfrac{y^2}{25} - \dfrac{x^2}{16} = 1$

14. $\dfrac{x^2}{169} - \dfrac{y^2}{144} = 1$

15. $\dfrac{(x+3)^2}{16} - \dfrac{(y-1)^2}{4} = 1$

16. $\dfrac{(x-1)^2}{2} - \dfrac{(y+3)^2}{4} = 1$

In Exercises 17–22, graph the hyperbola using a function grapher.

17. $\dfrac{x^2}{36} - \dfrac{y^2}{16} = 1$

18. $\dfrac{y^2}{64} - \dfrac{x^2}{16} = 1$

19. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

20. $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

21. $\dfrac{x^2}{4} - \dfrac{(y-3)^2}{5} = 1$

22. $\dfrac{(y-3)^2}{9} - \dfrac{(x+2)^2}{4} = 1$

In Exercises 23–38, find an equation in standard form for the hyperbola that satisfies the given conditions.

23. Foci $(\pm 3, 0)$, transverse axis length 4

24. Foci $(0, \pm 3)$, transverse axis length 4

25. Foci $(0, \pm 15)$, transverse axis length 8

26. Foci $(\pm 5, 0)$, transverse axis length 3

27. Center at $(0, 0)$, $a = 5$, $e = 2$, horizontal focal axis

28. Center at $(0, 0)$, $a = 4$, $e = 3/2$, vertical focal axis

29. Center at $(0, 0)$, $b = 5$, $e = 13/12$, vertical focal axis

30. Center at $(0, 0)$, $c = 6$, $e = 2$, horizontal focal axis

31. Transverse axis endpoints $(2, 3)$ and $(2, -1)$, conjugate axis length 6

32. Transverse axis endpoints $(5, 3)$ and $(-7, 3)$, conjugate axis length 10

33. Transverse axis endpoints $(-1, 3)$ and $(5, 3)$, slope of one asymptote 4/3

34. Transverse axis endpoints $(-2, -2)$ and $(-2, 7)$, slope of one asymptote 4/3

35. Foci $(-4, 2)$ and $(2, 2)$, transverse axis endpoints $(-3, 2)$ and $(1, 2)$

36. Foci $(-3, -11)$ and $(-3, 0)$, transverse axis endpoints $(-3, -9)$ and $(-3, -2)$

37. Center at $(-3, 6)$, $a = 5$, $e = 2$, vertical focal axis

38. Center at $(1, -4)$, $c = 6$, $e = 2$, horizontal focal axis

In Exercises 39–42, find the center, vertices, and the foci of the hyperbola.

39. $\dfrac{(x+1)^2}{144} - \dfrac{(y-2)^2}{25} = 1$

40. $\dfrac{(x+4)^2}{12} - \dfrac{(y+6)^2}{13} = 1$

41. $\dfrac{(y+3)^2}{64} - \dfrac{(x-2)^2}{81} = 1$

42. $\dfrac{(y-1)^2}{25} - \dfrac{(x+5)^2}{11} = 1$

In Exercises 43–46, graph the hyperbola using a parametric grapher in Dot graphing mode.

43. $\dfrac{y^2}{25} - \dfrac{x^2}{4} = 1$

44. $\dfrac{x^2}{30} - \dfrac{y^2}{20} = 1$

45. $\dfrac{(x+3)^2}{12} - \dfrac{(y-6)^2}{5} = 1$

46. $\dfrac{(y+1)^2}{15} - \dfrac{(x-2)^2}{6} = 1$

In Exercises 47–50, graph the hyperbola, and find its vertices, foci, and eccentricity.

47. $4(y-1)^2 - 9(x-3)^2 = 36$

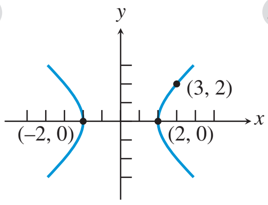
48. $4(x-2)^2 - 9(y+4)^2 = 1$

49. $9x^2 - 4y^2 - 36x + 8y - 4 = 0$

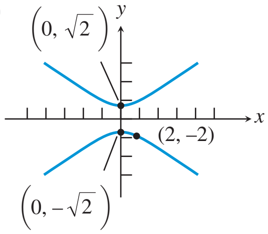
50. $25y^2 - 9x^2 - 50y - 54x - 281 = 0$

In Exercises 51 and 52, write an equation for the hyperbola.

51.



52.



53. Writing to Learn Prove that an equation for the hyperbola with center $(0, 0)$, foci $(0, \pm c)$, and semitransverse axis a is $y^2/a^2 - x^2/b^2 = 1$, where $c > a > 0$ and $b^2 = c^2 - a^2$. [*Hint*: Refer to derivation at the beginning of the section.]

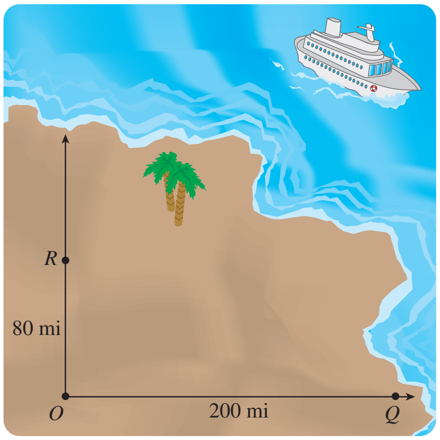
54. Degenerate Hyperbolas Graph the degenerate hyperbola.

(a) $\dfrac{x^2}{4} - y^2 = 0$ **(b)** $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 0$

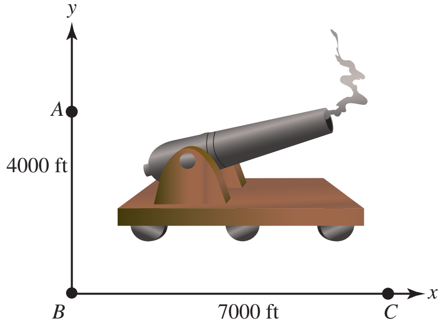
55. Rogue Comet A comet following a hyperbolic path about the Sun has a perihelion of 120 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 250 Gm from the Sun. Calculate a, b, c, and e. What are the coordinates of the center of the Sun if the center of the hyperbolic orbit is (0, 0) and the Sun lies on the positive x-axis?

56. Rogue Comet A comet following a hyperbolic path about the Sun has a perihelion of 140 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 405 Gm from the Sun. Calculate a, b, c, and e. What are the coordinates of the center of the Sun if the center of the hyperbolic orbit is (0, 0) and the Sun lies on the positive x-axis?

57. Long-Range Navigation Three LORAN radio transmitters are positioned as shown in the figure, with R due north of O and Q due east of O. The cruise ship *Princess Ann* receives simultaneous signals from the three transmitters. The signal from O arrives 323.27 μsec after the signal from R, and 646.53 μsec after the signal from Q. Determine the ship's bearing and distance from point O.



58. Gun Location Observers are located at positions A, B, and C with A due north of B. A cannon is located somewhere in the first quadrant as illustrated in the figure. A hears the sound of the cannon 2 sec before B, and C hears the sound 4 sec before B. Determine the bearing and distance of the cannon from point B. (Assume that sound travels at 1100 ft/sec.)



Group Activities In Exercises 59 and 60, solve the system of equations algebraically and support your answer graphically.

59. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

$x - \dfrac{2\sqrt{3}}{3}y = -2$

60. $\dfrac{x^2}{4} - y^2 = 1$

$x^2 + y^2 = 9$

61. Group Activity Consider the system of equations

$$\frac{x^2}{4} - \frac{y^2}{25} = 1$$
$$\frac{x^2}{25} + \frac{y^2}{4} = 1.$$

(a) Solve the system graphically.

(b) If you have access to a grapher that also does symbolic algebra, use it to find the the exact solutions to the system.

62. Writing to Learn Escape of the Unbound When NASA launches a space probe, the probe reaches a speed sufficient for it to become unbound from Earth and escape along a hyperbolic trajectory. Look up *escape speed* in an astronomy textbook or on the Internet, and write a paragraph in your own words about what you find.

Standardized Test Questions

63. True or False The distance from a focus of a hyperbola to the closer vertex is $a(e - 1)$, where a is the semitransverse axis and e is the eccentricity. Justify your answer.

64. True or False Unlike that for an ellipse, the Pythagorean relation for a hyperbola is the usual $a^2 + b^2 = c^2$. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve the problem.

65. Multiple Choice One focus of $x^2 - 4y^2 = 4$ is

(A) $(4, 0)$.

(B) $(\sqrt{5}, 0)$.

(C) $(2, 0)$.

(D) $(\sqrt{3}, 0)$.

(E) $(1, 0)$.

66. Multiple Choice The focal axis of

$$\frac{(x+5)^2}{9} - \frac{(y-6)^2}{16} = 1 \text{ is}$$

(A) $y = 2$.
(B) $y = 3$.
(C) $y = 4$.
(D) $y = 5$.
(E) $y = 6$.

67. Multiple Choice The center of $4x^2 - 12y^2 - 16x - 72y - 44 = 0$ is

(A) $(2, -2)$.
(B) $(2, -3)$.
(C) $(2, -4)$.
(D) $(2, -5)$.
(E) $(2, -6)$.

68. Multiple Choice The slopes of the asymptotes of the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ are

(A) ± 1.
(B) $\pm 3/2$.
(C) $\pm \sqrt{3}/2$.
(D) $\pm 2/3$.
(E) $\pm 4/3$.

Explorations

69. Constructing Points of a Hyperbola Use a geometry software package, such as *Cabri Geometry II*™, *The Geometer's Sketchpad*®, or a similar application on a handheld device, to carry out the following construction.

(a) Start by placing the coordinate axes in the construction window.

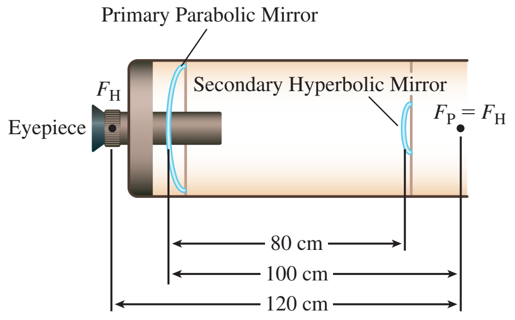
(b) Construct two points on the x-axis at $(\pm 5, 0)$ as the foci.

(c) Construct concentric circles of radii $r = 1, 2, 3, \ldots, 12$ centered at these two foci.

(d) Construct the points where these concentric circles meet and have a difference of radii of $2a = 6$, and overlay the conic that passes through these points if the software has a conic tool.

(e) Find the equation whose graph includes all of these points.

70. Cassegrain Telescope A Cassegrain telescope as described in the section has the dimensions shown in the figure. Find the standard form for the equation of the hyperbola centered at the origin with the x-axis as the focal axis.



Extending the Ideas

71. Prove that a nondegenerate graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is a hyperbola if $AC < 0$.

72. Writing to Learn The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0$$

is considered to be a degenerate hyperbola. Describe the graph. How is it like a full-fledged hyperbola, and how is it different?

73. Conjugate Hyperbolas The hyperbolas

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{and} \quad \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

obtained by switching the order of subtraction in their standard equations are **conjugate hyperbolas**. Prove that these hyperbolas have the same asymptotes and that the conjugate axis of each of these hyperbolas is the transverse axis of the other hyperbola.

74. Focal Width of a Hyperbola Prove that for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

if $x = c$, then $y = \pm b^2/a$. Why is it reasonable to define the **focal width** of such hyperbolas to be $2b^2/a$?

75. Writing to Learn Explain how the standard form equations for the conics are related to

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$