5.5 The Law of Sines

What you'll learn about

- Deriving the Law of Sines
- Solving Triangles (AAS, ASA)
- The Ambiguous Case (SSA)
- Applications

... and why

The Law of Sines is a powerful extension of the triangle congruence theorems of Euclidean geometry.

Deriving the Law of Sines

Recall from geometry that a triangle has six parts (three sides (S), three angles (A)), but that its size and shape can be completely determined by fixing only three of those parts, provided they are the right three. These threesomes that determine *triangle congruence* are known by their acronyms: AAS, ASA, SAS, and SSS. The other two acronyms represent matchups that don't quite work: AAA determines similarity only, while SSA does not even determine similarity.

With trigonometry we can find the other parts of the triangle once congruence is established. The tools we need are the Law of Sines and the Law of Cosines, the subjects of our last two trigonometric sections.

The **Law of Sines** states that the ratio of the sine of an angle to the length of its opposite side is the same for all three angles of any triangle.

Law of Sines

In any ΔABC with angles A, B, and C opposite sides a, b, and c, respectively, the following equation is true:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

The derivation of the Law of Sines refers to the two triangles in Figure 5.13, in each of which we have drawn an altitude to side c. Right triangle trigonometry applied to either of the triangles in Figure 5.13 tells us that

$$\sin A = \frac{h}{b}.$$

In the acute triangle on the top,

$$\sin B = \frac{h}{a},$$

while in the obtuse triangle on the bottom,

$$\sin(\pi - B) = \frac{h}{a}.$$

But $\sin(\pi - B) = \sin B$, so in either case

$$\sin B = \frac{h}{a}.$$

Solving for h in both equations yields $h = b \sin A = a \sin B$. The equation $b \sin A = a \sin B$ is equivalent to

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

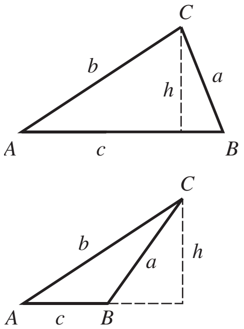


FIGURE 5.13 The Law of Sines.

If we were to draw an altitude to side a and repeat the same steps as above, we would reach the conclusion that

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

Putting the results together,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Solving Triangles (AAS, ASA)

Two angles and a side of a triangle, in any order, determine the size and shape of a triangle completely. Of course, two angles of a triangle determine the third, so we really get one of the missing three parts for free. We solve for the remaining two parts (the unknown sides) with the Law of Sines.

EXAMPLE 1 Solving a Triangle Given Two Angles and a Side

Solve ΔABC given that $\angle A = 36°$, $\angle B = 48°$, and $a = 8$. (See Figure 5.14.)

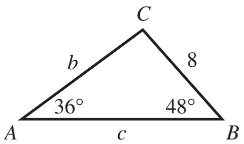


FIGURE 5.14 A triangle determined by AAS. (Example 1)

SOLUTION First, we note that $\angle C = 180° - 36° - 48° = 96°$.

We then apply the Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{and} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 36°}{8} = \frac{\sin 48°}{b} \qquad \frac{\sin 36°}{8} = \frac{\sin 96°}{c}$$

$$b = \frac{8 \sin 48°}{\sin 36°} \qquad c = \frac{8 \sin 96°}{\sin 36°}$$

$$b \approx 10.115 \qquad c \approx 13.536$$

The six parts of the triangle are:

$$\angle A = 36° \qquad a = 8$$
$$\angle B = 48° \qquad b \approx 10.115$$
$$\angle C = 96° \qquad c \approx 13.536$$

Now try Exercise 1.

The Ambiguous Case (SSA)

While two angles and a side of a triangle are always sufficient to determine its size and shape, the same cannot be said for two sides and an angle. Perhaps unexpectedly, it depends on where that angle is. If the angle is included between the two sides (the SAS case), then the triangle is uniquely determined up to congruence. If the angle is opposite one of the sides (the SSA case), then there might be one, two, or zero triangles determined.

Solving a triangle in the SAS case involves the Law of Cosines and will be handled in the next section. Solving a triangle in the SSA case is done with the Law of Sines, but with an eye toward the possibilities, as seen in the following Exploration.

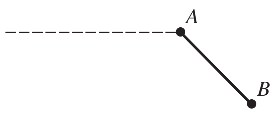


FIGURE 5.15 The diagram for part 1. (Exploration 1)

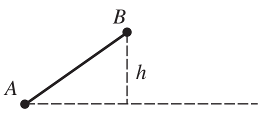


FIGURE 5.16 The diagram for part 2. (Exploration 1)

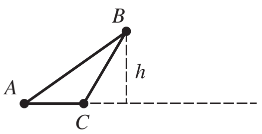


FIGURE 5.17 The diagram for parts 3–5. (Exploration 1)

EXPLORATION 1 Determining the Number of Triangles

We wish to construct ΔABC given angle A, side AB, and side BC.

1. Suppose $\angle A$ is obtuse and that side AB is as shown in Figure 5.15. To complete the triangle, side BC must determine a point on the dotted horizontal line (which extends infinitely to the left). Explain from the picture why a *unique* triangle ΔABC is determined if $BC > AB$, but *no* triangle is determined if $BC \leq AB$.
2. Suppose $\angle A$ is acute and that side AB is as shown in Figure 5.16. To complete the triangle, side BC must determine a point on the dotted horizontal line (which extends infinitely to the right). Explain from the picture why a *unique* triangle ΔABC is determined if $BC = h$, but *no* triangle is determined if $BC < h$.
3. Suppose $\angle A$ is acute and that side AB is as shown in Figure 5.17. If $AB > BC > h$, then we can form a triangle as shown. Find a *second* point C on the dotted horizontal line that gives a side BC of the same length, but determines a different triangle. (This is the "ambiguous case.")
4. Explain why $\sin C$ is the same in both triangles in the ambiguous case. (This is why the Law of Sines is also ambiguous in this case.)
5. Explain from Figure 5.17 why a *unique* triangle is determined if $BC \geq AB$.

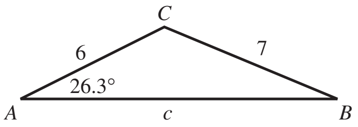


FIGURE 5.18 A triangle determined by SSA. (Example 2)

Now that we know what can happen, let us try the algebra.

EXAMPLE 2 Solving a Triangle Given Two Sides and an Angle

Solve ΔABC given that $a = 7$, $b = 6$, and $\angle A = 26.3°$. (See Figure 5.18.)

SOLUTION By drawing a reasonable sketch (Figure 5.18), we can assure ourselves that this is not the ambiguous case. (In fact, this is the case described in step 5 of Exploration 1.)

Begin by solving for the *acute* angle B, using the Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b}. \qquad \text{Law of Sines}$$

$$\frac{\sin 26.3°}{7} = \frac{\sin B}{6}$$

$$\sin B = \frac{6 \sin 26.3°}{7}$$

$$B = \sin^{-1}\left(\frac{6 \sin 26.3°}{7}\right)$$

$$B = 22.3° \qquad \text{Round to match accuracy of given angle.}$$

Then, find the obtuse angle C by subtraction:

$$\begin{aligned} C &= 180° - 26.3° - 22.3° \\ &= 131.4° \end{aligned}$$

Finally, find side c:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 26.3°}{7} = \frac{\sin 131.4°}{c}$$

$$c = \frac{7 \sin 131.4°}{\sin 26.3°}$$

$$c \approx 11.9$$

The six parts of the triangle are:

$$\begin{aligned} \angle A &= 26.3° & a &= 7 \\ \angle B &= 22.3° & b &= 6 \\ \angle C &= 131.4° & c &\approx 11.9 \end{aligned}$$

Now try Exercise 9.

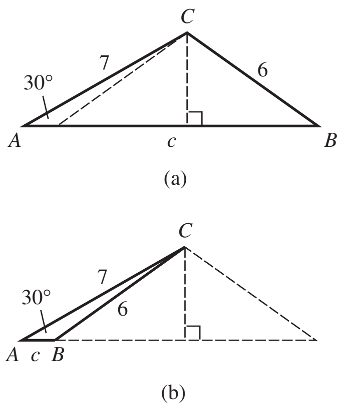


FIGURE 5.19 Two triangles determined by the same SSA values. (Example 3)

EXAMPLE 3 Handling the Ambiguous Case

Solve ΔABC given that $a = 6$, $b = 7$, and $\angle A = 30°$.

SOLUTION By drawing a reasonable sketch (Figure 5.19), we see that two triangles are possible with the given information. We keep this in mind as we proceed.

We begin by using the Law of Sines to find angle B.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$

$$\frac{\sin 30°}{6} = \frac{\sin B}{7}$$

$$\sin B = \frac{7 \sin 30°}{6}$$

$$B = \sin^{-1}\left(\frac{7 \sin 30°}{6}\right)$$

$$B = 35.7° \qquad \text{Round to match accuracy of given angle.}$$

Notice that the calculator gave us one value for B, not two. That is because we used the *function* $\sin^{-1}$, which cannot give two output values for the same input value. Indeed, the function $\sin^{-1}$ will *never give an obtuse angle*, which is why we chose to start with the acute angle in Example 2. In this case, the calculator has found the angle B shown in Figure 5.19a.

Find the obtuse angle C by subtraction:

$$C = 180° - 30.0° - 35.7° = 114.3°$$

Finally, find side c:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 30.0°}{6} = \frac{\sin 114.3°}{c}$$

$$c = \frac{6 \sin 114.3°}{\sin 30°}$$

$$c \approx 10.9$$

So, under the assumption that angle B is *acute* (see Figure 5.19a), the six parts of the triangle are:

$$\angle A = 30.0° \qquad a = 6$$
$$\angle B = 35.7° \qquad b = 7$$
$$\angle C = 114.3° \qquad c \approx 10.9$$

If angle B is *obtuse*, then we can see from Figure 5.19b that it has measure $180° - 35.7° = 144.3°$.

By subtraction, the acute angle $C = 180° - 30.0° - 144.3° = 5.7°$. We then recompute c:

$$c = \frac{6 \sin 5.7°}{\sin 30°} \approx 1.2$$ Substitute 5.7° for 114.3° in earlier computation.

So, under the assumption that angle B is *obtuse* (see Figure 5.19b), the six parts of the triangle are:

$$\angle A = 30.0° \qquad a = 6$$
$$\angle B = 144.3° \qquad b = 7$$
$$\angle C = 5.7° \qquad c \approx 1.2$$

Now try Exercise 19.

Applications

Many problems involving angles and distances can be solved by superimposing a triangle onto the situation and solving the triangle.

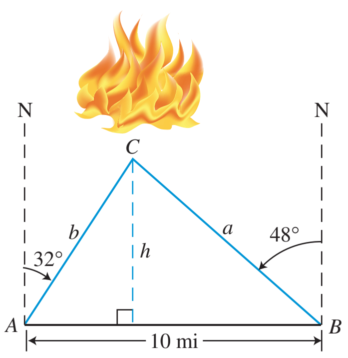


FIGURE 5.20 Determining the location of a fire. (Example 4)

EXAMPLE 4 Locating a Fire

Forest Ranger Chris Johnson at ranger station A sights a fire in the direction 32° east of north. Ranger Rick Thorpe at ranger station B, 10 miles due east of A, sights the same fire on a line 48° west of north. Find the distance from each ranger station to the fire.

SOLUTION Let C represent the location of the fire. A sketch (Figure 5.20) shows the superimposed triangle, ΔABC, in which angles A and B and their included side (AB) are known. This is a setup for the Law of Sines.

(continued)

Note that $\angle A = 90° - 32° = 58°$ and $\angle B = 90° - 48° = 42°$. By subtraction, we find that $\angle C = 180° - 58° - 42° = 80°$.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{and} \quad \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$\frac{\sin 58°}{a} = \frac{\sin 80°}{10} \qquad \frac{\sin 42°}{b} = \frac{\sin 80°}{10}$$

$$a = \frac{10 \sin 58°}{\sin 80°} \qquad b = \frac{10 \sin 42°}{\sin 80°}$$

$$a \approx 8.6 \qquad b \approx 6.8$$

The fire is about 6.8 miles from ranger station A and about 8.6 miles from ranger station B. ***Now try Exercise 45.***

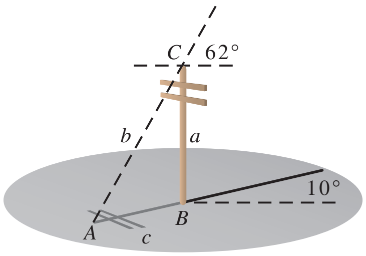


FIGURE 5.21 A telephone pole on a slope. (Example 5)

EXAMPLE 5 Finding the Height of a Pole

A road slopes 10° above the horizontal, and a vertical telephone pole stands beside the road. The angle of elevation of the Sun is 62°, and the pole casts a 14.5-foot shadow downhill along the road. Find the height of the telephone pole.

SOLUTION This is an interesting variation on a typical application of right triangle trigonometry. The slope of the road eliminates the convenient right angle, but we can still solve the problem by solving a triangle.

Figure 5.21 shows the superimposed triangle, ΔABC. A little preliminary geometry is required to find the measure of angles A and C. Due to the slope of the road, angle A is 10° less than the angle of elevation of the Sun and angle B is 10° more than a right angle. That is,

$$\angle A = 62° - 10° = 52°$$
$$\angle B = 90° + 10° = 100°$$
$$\angle C = 180° - 52° - 100° = 28°$$

Therefore,

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$\frac{\sin 52°}{a} = \frac{\sin 28°}{14.5}$$

$$a = \frac{14.5 \sin 52°}{\sin 28°}$$

$$a \approx 24.3 \qquad \text{Round to match accuracy of input.}$$

The pole is approximately 24.3 feet high. ***Now try Exercise 39.***

QUICK REVIEW 5.5 *(For help, go to Sections 4.2 and 4.7.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, solve the equation $a/b = c/d$ for the given variable.

1. a **2.** b **3.** c **4.** d

In Exercises 5 and 6, evaluate the expression.

5. $\dfrac{7 \sin 48°}{\sin 23°}$ **6.** $\dfrac{9 \sin 121°}{\sin 14°}$

In Exercises 7–10, solve for the angle x.

7. $\sin x = 0.3, \quad 0° < x < 90°$

8. $\sin x = 0.3, \quad 90° < x < 180°$

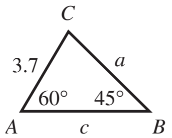
9. $\sin x = -0.7, \quad 180° < x < 270°$

10. $\sin x = -0.7, \quad 270° < x < 360°$

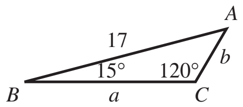
SECTION 5.5 EXERCISES

In Exercises 1–4, solve the triangle.

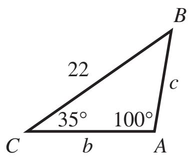
1.



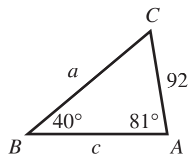
2.



3.



4.



In Exercises 5–8, solve the triangle.

5. $A = 40°, \quad B = 30°, \quad b = 10$

6. $A = 50°, \quad B = 62°, \quad a = 4$

7. $A = 33°, \quad B = 70°, \quad b = 7$

8. $B = 16°, \quad C = 103°, \quad c = 12$

In Exercises 9–12, solve the triangle.

9. $A = 32°, \quad a = 17, \quad b = 11$

10. $A = 49°, \quad a = 32, \quad b = 28$

11. $B = 70°, \quad b = 14, \quad c = 9$

12. $C = 103°, \quad b = 46, \quad c = 61$

In Exercises 13–18, state whether the given measurements determine zero, one, or two triangles.

13. $A = 36°, \quad a = 2, \quad b = 7$

14. $B = 82°, \quad b = 17, \quad c = 15$

15. $C = 36°, \quad a = 17, \quad c = 16$

16. $A = 73°, \quad a = 24, \quad b = 28$

17. $C = 30°, \quad a = 18, \quad c = 9$

18. $B = 88°, \quad b = 14, \quad c = 62$

In Exercises 19–22, two triangles can be formed using the given measurements. Solve both triangles.

19. $A = 64°, \quad a = 16, \quad b = 17$

20. $B = 38°, \quad b = 21, \quad c = 25$

21. $C = 68°, \quad a = 19, \quad c = 18$

22. $B = 57°, \quad a = 11, \quad b = 10$

23. Determine the values of b that will produce the given number of triangles if $a = 10$ and $B = 42°$.

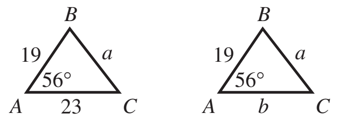
(a) Two triangles **(b)** One triangle **(c)** Zero triangles

24. Determine the values of c that will produce the given number of triangles if $b = 12$ and $C = 53°$.

(a) Two triangles **(b)** One triangle **(c)** Zero triangles

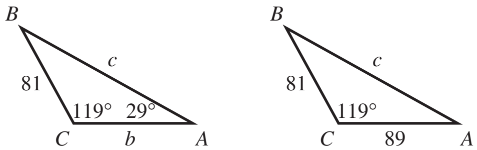
In Exercises 25 and 26, decide whether the triangle can be solved using the Law of Sines. If so, solve it. If not, explain why not.

25.



(a) (b)

26.



(a) (b)

In Exercises 27–36, respond in one of the following ways:

(a) State, "Cannot be solved with the Law of Sines."

(b) State, "No triangle is formed."

(c) Solve the triangle.

27. $A = 61°, \quad a = 8, \quad b = 21$

28. $B = 47°, \quad a = 8, \quad b = 21$

29. $A = 136°, \quad a = 15, \quad b = 28$

30. $C = 115°, \quad b = 12, \quad c = 7$

31. $B = 42°, \quad c = 18, \quad C = 39°$

32. $A = 19°, \quad b = 22, \quad B = 47°$

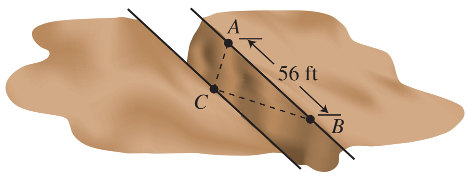
33. $C = 75°, \quad b = 49, \quad c = 48$

34. $A = 54°, \quad a = 13, \quad b = 15$

35. $B = 31°, \quad a = 8, \quad c = 11$

36. $C = 65°, \quad a = 19, \quad b = 22$

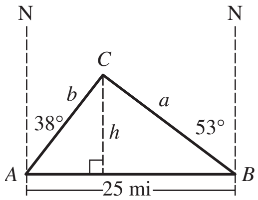
37. Surveying a Canyon Two markers A and B on the same side of a canyon rim are 56 ft apart. A third marker C, located across the rim, is positioned so that $\angle BAC = 72°$ and $\angle ABC = 53°$.



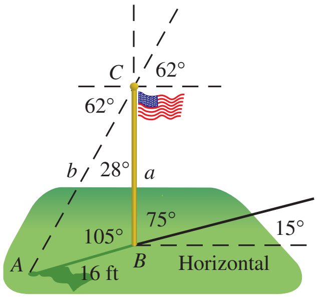
(a) Find the distance between C and A.

(b) Find the distance between the two canyon rims. (Assume they are parallel.)

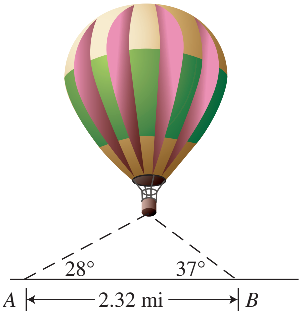
38. Weather Forecasting Two meteorologists are 25 mi apart located on an east-west road. The meteorologist at point A sights a tornado 38° east of north. The meteorologist at point B sights the same tornado 53° west of north. Find the distance from each meteorologist to the tornado. Also find the distance between the tornado and the road.



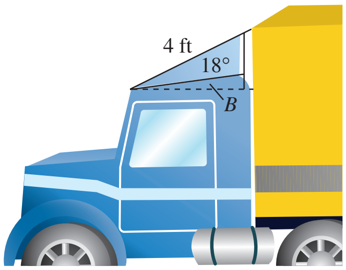
39. Engineering Design A vertical flagpole stands beside a road that slopes at an angle of 15° with the horizontal. When the angle of elevation of the Sun is 62°, the flagpole casts a 16-ft shadow downhill along the road. Find the height of the flagpole.



40. Altitude Observers 2.32 mi apart see a hot-air balloon directly between them but at the angles of elevation shown in the figure. Find the altitude of the balloon.



41. Reducing Air Resistance A 4-ft airfoil attached to the cab of a truck reduces wind resistance. If the angle between the airfoil and the cab top is 18° and angle B is 10°, find the length of a vertical brace positioned as shown in the figure.

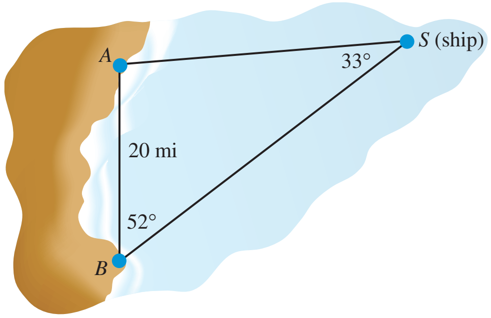


42. Group Activity Ferris Wheel Design A Ferris wheel has 16 evenly spaced cars. The distance between adjacent chairs is 15.5 ft. Find the radius of the wheel (to the nearest 0.1 ft).

43. Finding Height Two observers are 600 ft apart on opposite sides of a flagpole. The angles of elevation from the observers to the top of the pole are 19° and 21°. Find the height of the flagpole.

44. Finding Height Two observers are 400 ft apart on opposite sides of a tree. The angles of elevation from the observers to the top of the tree are 15° and 20°. Find the height of the tree.

45. Finding Distance Two lighthouses A and B are known to be exactly 20 mi apart on a north-south line. A ship's captain at S measures $\angle ASB$ to be 33°. A radio operator at B measures $\angle ABS$ to be 52°. Find the distance from the ship to each lighthouse.

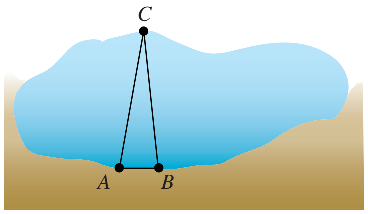


46. Using Measurement Data A geometry class is divided into ten teams, each of which is given a yardstick and a protractor to find the distance from a point A on the edge of a pond to a tree at a point C on the opposite shore. After they mark points A and B with stakes, each team uses a protractor to measure angles A and B and a yardstick to measure distance AB. Their measurements are given in the table.

A	B	AB
79°	84°	26′ 4″
81°	82°	25′ 5″
79°	83°	26′ 0″
80°	87°	26′ 1″
79°	87°	25′ 11″

A	B	AB
83°	84°	25′ 3″
82°	82°	26′ 5″
78°	85°	25′ 8″
77°	83°	26′ 4″
79°	82°	25′ 7″

Use the data to find the class's best estimate for the distance AC.



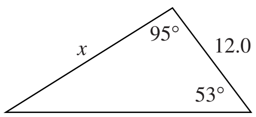
Standardized Test Questions

47. True or False The ratio of the sines of any two angles in a triangle equals the ratio of the lengths of their opposite sides. Justify your answer.

48. True or False The perimeter of a triangle with two 10-inch sides and two 40° angles is greater than 36. Justify your answer.

You may use a graphing calculator when answering these questions.

49. Multiple Choice The length x in the triangle shown at the right is



(A) 8.6. **(B)** 15.0. **(C)** 18.1.

(D) 19.2. **(E)** 22.6.

50. Multiple Choice Which of the following three triangle parts do not necessarily determine the other three parts?

(A) AAS **(B)** ASA **(C)** SAS

(D) SSA **(E)** SSS

51. Multiple Choice The shortest side of a triangle with angles 50°, 60°, and 70° has length 9.0. What is the length of the longest side?

(A) 11.0 **(B)** 11.5 **(C)** 12.0

(D) 12.5 **(E)** 13.0

52. Multiple Choice How many noncongruent triangles ABC can be formed if $AB = 5$, $A = 60°$, and $BC = 8$?

(A) None **(B)** One **(C)** Two

(D) Three **(E)** Infinitely many

Explorations

53. Writing to Learn

(a) Show that there are infinitely many triangles with AAA given if the sum of the three positive angles is 180°.

(b) Give three examples of triangles where $A = 30°$, $B = 60°$, and $C = 90°$.

(c) Give three examples where $A = B = C = 60°$.

54. Use the Law of Sines and the cofunction identities to derive the following formulas from right triangle trigonometry:

(a) $\sin A = \dfrac{opp}{hyp}$ **(b)** $\cos A = \dfrac{adj}{hyp}$ **(c)** $\tan A = \dfrac{opp}{adj}$

55. Wrapping up Exploration 1 Refer to Figures 5.16 and 5.17 in Exploration 1 of this section.

(a) Express h in terms of angle A and length AB.

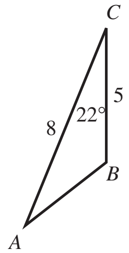
(b) In terms of the given angle A and the given length AB, state the conditions on length BC that will result in no triangle being formed.

(c) In terms of the given angle A and the given length AB, state the conditions on length BC that will result in a unique triangle being formed.

(d) In terms of the given angle A and the given length AB, state the conditions on length BC that will result in two possible triangles being formed.

Extending the Ideas

56. Solve this triangle assuming that $\angle B$ is obtuse. [*Hint*: Draw a perpendicular from A to the line through B and C.]



57. Pilot Calculations Towers A and B are known to be 4.1 mi apart on level ground. A pilot measures the angles of depression to the towers to be 36.5° and 25°, respectively, as shown in the figure. Find distances AC and BC and the height of the airplane.