5.2 Proving Trigonometric Identities

What you'll learn about

- A Proof Strategy
- Proving Identities
- Disproving Non-Identities
- Identities in Calculus

... and why

Proving identities gives you excellent insights into the way mathematical proofs are constructed.

A Proof Strategy

We now arrive at the best opportunity in the precalculus curriculum for you to try your hand at constructing analytic proofs: trigonometric identities. Some are easy and some can be quite challenging, but in every case the *identity itself* frames your work with a beginning and an ending. The proof consists of filling in what lies between.

The strategy for proving an identity is very different from the strategy for solving an equation, most notably in the very first step. Usually the first step in solving an equation is to write down the equation. If you do this with an identity, however, you will have a beginning and an ending—with no proof in between! With an identity, you begin by writing down *one function* and end by writing down *the other*. Example 1 will illustrate what we mean.

EXAMPLE 1 Proving an Algebraic Identity

Prove the algebraic identity $\dfrac{x^2-1}{x-1} - \dfrac{x^2-1}{x+1} = 2.$

SOLUTION We prove this identity by showing a sequence of expressions, each one easily seen to be equivalent to its preceding expression:

$$
\begin{aligned}
\frac{x^2-1}{x-1} - \frac{x^2-1}{x+1} &= \frac{(x+1)(x-1)}{x-1} - \frac{(x+1)(x-1)}{x+1} && \text{Factoring difference of squares}\\
&= (x+1)\left(\frac{x-1}{x-1}\right) - (x-1)\left(\frac{x+1}{x+1}\right) && \text{Algebraic manipulation}\\
&= (x+1)(1) - (x-1)(1) && \text{Reducing fractions}\\
&= x+1-x+1 && \text{Algebraic manipulation}\\
&= 2
\end{aligned}
$$

Notice that the first thing we wrote down was the expression on the left-hand side (LHS), and the last thing we wrote down was the expression on the right-hand side (RHS). The proof would have been just as legitimate going from RHS to LHS, but it is more natural to move from the more complicated side to the less complicated side. Incidentally, the margin notes on the right, called "floaters," are included here for instructional purposes and are not really part of the proof. A good proof should consist of steps for which a knowledgeable reader could readily supply the floaters.

Now try Exercise 1.

These, then, are our first general strategies for proving an identity:

General Strategies I

1. The proof begins with the expression on one side of the identity.
2. The proof ends with the expression on the other side.
3. The proof in between consists of showing a sequence of expressions, each one easily seen to be equivalent to its preceding expression.

Proving Identities

Trigonometric identity proofs follow General Strategies I. We are told that two expressions are equal, and the object is to prove that they are equal. We do this by changing one expression into the other by a series of intermediate steps that follow the important rule that *every intermediate step yields an expression that is equivalent to the first.*

The changes at every step are accomplished by algebraic manipulations or identities, but the manipulations or identities should be sufficiently obvious as to require no additional justification. Since "obvious" is often in the eye of the beholder, it is usually safer to err on the side of including too many steps than too few.

By working through several examples, we try to give you a sense for what is appropriate as we illustrate some of the algebraic tools that you have at your disposal.

EXAMPLE 2 Proving an Identity

Prove the identity: $\tan x + \cot x = \sec x \csc x$.

SOLUTION We begin by deciding whether to start with the expression on the right or the left. It is usually best to start with the more complicated expression, as it is easier to proceed from the complex toward the simple than to go in the other direction. The expression on the left is slightly more complicated because it involves two terms.

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} && \text{Basic identities} \\
&= \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\sin x} + \frac{\cos x}{\sin x} \cdot \frac{\cos x}{\cos x} && \text{Setting up common denominator} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x} \\
&= \frac{1}{\cos x \cdot \sin x} && \text{Pythagorean identity} \\
&= \frac{1}{\cos x} \cdot \frac{1}{\sin x} && \text{(A step you could choose to omit)} \\
&= \sec x \csc x && \text{Basic identities}
\end{aligned}$$

(Remember that the "floaters" are not really part of the proof.)

Now try Exercise 13.

The preceding example illustrates three general strategies that are often useful in proving trigonometric identities.

General Strategies II

1. Begin with the more complicated expression and work toward the less complicated expression.
2. If no other move suggests itself, convert the entire expression to one involving sines and cosines.
3. Combine fractions by combining them over a common denominator.

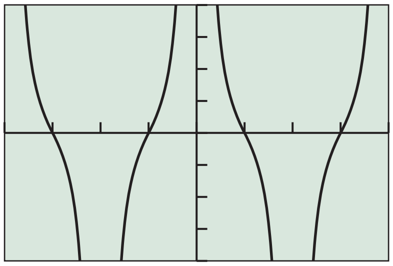
EXAMPLE 3 Identifying and Proving an Identity

Match the function

$$f(x) = \frac{1}{\sec x - 1} + \frac{1}{\sec x + 1}$$

with one of the following. Then confirm the match with a proof.

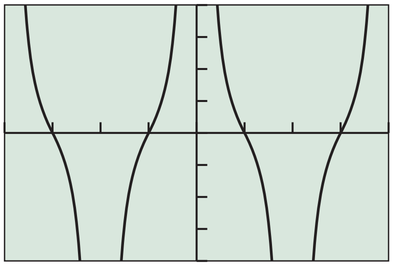
(i) $2 \cot x \csc x$ **(ii)** $\dfrac{1}{\sec x}$



$[-2\pi, 2\pi]$ by $[-4, 4]$

$f(x) = \dfrac{1}{\sec x - 1} + \dfrac{1}{\sec x + 1}$

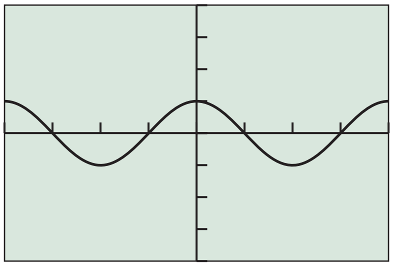
(a)



$[-2\pi, 2\pi]$ by $[-4, 4]$

$y = 2 \cot x \csc x$

(b)



$[-2\pi, 2\pi]$ by $[-4, 4]$

$y = \dfrac{1}{\sec x}$

(c)

FIGURE 5.8 A grapher can be useful for identifying possible identities. (Example 3)

SOLUTION Figures 5.8a, b, and c show the graphs of the functions $y = f(x)$, $y = 2 \cot x \csc x$, and $y = 1/\sec x$, respectively. The graphs in (a) and (c) show that $f(x)$ is not equal to the expression in (ii). From the graphs in (a) and (b), it appears that $f(x)$ is equal to the expression in (i). To confirm, we begin with the expression for $f(x)$.

$$\frac{1}{\sec x - 1} + \frac{1}{\sec x + 1}$$

$$= \frac{\sec x + 1}{(\sec x - 1)(\sec x + 1)} + \frac{\sec x - 1}{(\sec x - 1)(\sec x + 1)} \qquad \text{Common denominator}$$

$$= \frac{\sec x + 1 + \sec x - 1}{\sec^2 x - 1}$$

$$= \frac{2 \sec x}{\tan^2 x} \qquad \text{Pythagorean identity}$$

$$= \frac{2}{\cos x} \cdot \frac{\cos^2 x}{\sin^2 x} \qquad \text{Basic identities}$$

$$= \frac{2 \cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= 2 \cot x \csc x$$

Now try Exercise 55.

The next example illustrates how the algebraic identity $(a + b)(a - b) = a^2 - b^2$ can be used to set up a Pythagorean substitution.

EXAMPLE 4 Setting up a Difference of Squares

Prove the identity: $\cos t/(1 - \sin t) = (1 + \sin t)/\cos t$.

SOLUTION The left-hand expression is slightly more complicated, as we can handle extra terms in a numerator more easily than in a denominator. So we begin with the left.

$$\frac{\cos t}{1 - \sin t} = \frac{\cos t}{1 - \sin t} \cdot \frac{1 + \sin t}{1 + \sin t} \qquad \text{Setting up a difference of squares}$$

$$= \frac{(\cos t)(1 + \sin t)}{1 - \sin^2 t}$$

$$= \frac{(\cos t)(1 + \sin t)}{\cos^2 t} \qquad \text{Pythagorean identity}$$

$$= \frac{1 + \sin t}{\cos t}$$

Now try Exercise 39.

Notice that we kept $(\cos t)(1 + \sin t)$ in factored form in the hope that we could eventually eliminate the factor $\cos t$ and be left with the numerator we need. It is always a good idea to keep an eye on the "target" expression toward which your proof is aimed.

General Strategies III

1. Use the algebraic identity $(a + b)(a - b) = a^2 - b^2$ to set up applications of the Pythagorean identities.
2. Always be mindful of the "target" expression, and favor manipulations that bring you closer to your goal.

In more complicated identities (as in word ladders) it is sometimes helpful to see if both sides can be manipulated toward a common intermediate expression. The proof can then be reconstructed in a single path.

EXAMPLE 5 Working from Both Sides

Prove the identity: $\cot^2 u/(1 + \csc u) = (\cot u)(\sec u - \tan u)$.

SOLUTION Both sides are fairly complicated, but the left-hand side looks as though it needs more work. We start on the left.

$$
\begin{aligned}
\frac{\cot^2 u}{1+\csc u} &= \frac{\csc^2 u - 1}{1 + \csc u} && \text{Pythagorean identity} \\
&= \frac{(\csc u + 1)(\csc u - 1)}{\csc u + 1} && \text{Factor} \\
&= \csc u - 1
\end{aligned}
$$

At this point it is not clear how we can get from this expression to the one on the right-hand side of our identity. However, we now have reason to believe that the right-hand side must simplify to $\csc u - 1$, so we try simplifying the right-hand side.

$$
\begin{aligned}
(\cot u)(\sec u - \tan u) &= \left(\frac{\cos u}{\sin u}\right)\left(\frac{1}{\cos u} - \frac{\sin u}{\cos u}\right) && \text{Basic identities} \\
&= \frac{1}{\sin u} - 1 && \text{Distribute the product.} \\
&= \csc u - 1
\end{aligned}
$$

Now we can reconstruct the proof by going through $\csc u - 1$ as an intermediate step.

$$
\begin{aligned}
\frac{\cot^2 u}{1+\csc u} &= \frac{\csc^2 u - 1}{1 + \csc u} \\
&= \frac{(\csc u + 1)(\csc u - 1)}{\csc u + 1} \\
&= \csc u - 1 && \text{Intermediate step} \\
&= \frac{1}{\sin u} - 1 \\
&= \left(\frac{\cos u}{\sin u}\right)\left(\frac{1}{\cos u} - \frac{\sin u}{\cos u}\right) \\
&= (\cot u)(\sec u - \tan u)
\end{aligned}
$$

Now try Exercise 41.

Disproving Non-Identities

Obviously, not every equation involving trigonometric expressions is an identity. How can we spot a non-identity before embarking on a futile attempt at a proof? Try the following exploration.

EXPLORATION 1 Confirming a Non-Identity

Prove or disprove that this is an identity: $\cos 2x = 2 \cos x$.

1. Graph $y = \cos 2x$ and $y = 2 \cos x$ in the same window. Interpret the graphs to make a conclusion about whether or not the equation is an identity.
2. With the help of the graph, find a value of x for which $\cos 2x \neq 2 \cos x$.
3. Does the existence of the x-value in part (2) *prove* that the equation is *not* an identity?

4. Graph $y = \cos 2x$ and $y = \cos^2 x - \sin^2 x$ in the same window. Interpret the graphs to make a conclusion about whether or not $\cos 2x = \cos^2 x - \sin^2 x$ is an identity.
5. Do the graphs in part (4) *prove* that $\cos 2x = \cos^2 x - \sin^2 x$ is an identity? Explain your answer.

Exploration 1 suggests that we can use graphers to help confirm a *non-identity*, since we only have to produce a single value of x for which the two compared expressions are defined but unequal. On the other hand, we cannot use graphers to prove that an equation *is* an identity, since, for example, the graphers can never prove that two irrational numbers are equal. Also, graphers cannot show behavior over infinite domains.

Identities in Calculus

In most calculus problems where identities play a role, the object is to make a complicated expression simpler for the sake of computational ease. Occasionally it is actually necessary to make a *simple* expression *more complicated* for the sake of computational ease. Each of the following identities (just a sampling of many) represents a useful substitution in calculus wherein the expression on the right is simpler to deal with (even though it does not look that way). We prove one of these identities in Example 6 and leave the rest for the exercises or for future sections.

1. $\cos^3 x = (1 - \sin^2 x)(\cos x)$
2. $\sec^4 x = (1 + \tan^2 x)(\sec^2 x)$
3. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$
4. $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$
5. $\sin^5 x = (1 - 2\cos^2 x + \cos^4 x)(\sin x)$
6. $\sin^2 x \cos^5 x = (\sin^2 x - 2\sin^4 x + \sin^6 x)(\cos x)$

EXAMPLE 6 Proving an Identity Useful in Calculus

Prove the following identity:

$$\sin^2 x \cos^5 x = (\sin^2 x - 2\sin^4 x + \sin^6 x)(\cos x).$$

SOLUTION We begin with the expression on the left.

$$\begin{aligned}
\sin^2 x \cos^5 x &= \sin^2 x \cos^4 x \cos x \\
&= (\sin^2 x)(\cos^2 x)^2 (\cos x) \\
&= (\sin^2 x)(1 - \sin^2 x)^2 (\cos x) \\
&= (\sin^2 x)(1 - 2\sin^2 x + \sin^4 x)(\cos x) \\
&= (\sin^2 x - 2\sin^4 x + \sin^6 x)(\cos x)
\end{aligned}$$

Now try Exercise 51.

QUICK REVIEW 5.2 *(For help, go to Section 5.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–6, write the expression in terms of sines and cosines only. Express your answer as a single fraction.

1. $\csc x + \sec x$

2. $\tan x + \cot x$

3. $\cos x \csc x + \sin x \sec x$

4. $\sin \theta \cot \theta - \cos \theta \tan \theta$

5. $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x}$

6. $\dfrac{\sec \alpha}{\cos \alpha} - \dfrac{\sin \alpha}{\sec \alpha \cos^2 \alpha}$

In Exercises 7–12, determine whether or not the equation is an identity. If not, find a single value of x for which the two expressions are not equal.

7. $\sqrt{x^2} = x$

8. $\sqrt[3]{x^3} = x$

9. $\sqrt{1 - \cos^2 x} = \sin x$

10. $\sqrt{\sec^2 x - 1} = \tan x$

11. $\ln \dfrac{1}{x} = -\ln x$

12. $\ln x^2 = 2 \ln x$

SECTION 5.2 EXERCISES

In Exercises 1–4, prove the algebraic identity by starting with the LHS expression and supplying a sequence of equivalent expressions that ends with the RHS expression.

1. $\dfrac{x^3 - x^2}{x} - (x - 1)(x + 1) = 1 - x$

2. $\dfrac{1}{x} - \dfrac{1}{2} = \dfrac{2 - x}{2x}$

3. $\dfrac{x^2 - 4}{x - 2} - \dfrac{x^2 - 9}{x + 3} = 5$

4. $(x - 1)(x + 2) - (x + 1)(x - 2) = 2x$

In Exercises 5–10, tell whether or not $f(x) = \sin x$ is an identity.

5. $f(x) = \dfrac{\sin^2 x + \cos^2 x}{\csc x}$

6. $f(x) = \dfrac{\tan x}{\sec x}$

7. $f(x) = \cos x \cdot \cot x$

8. $f(x) = \cos (x - \pi/2)$

9. $f(x) = (\sin^3 x)(1 + \cot^2 x)$

10. $f(x) = \dfrac{\sin 2x}{2}$

In Exercises 11–51, prove the identity.

11. $(\cos x)(\tan x + \sin x \cot x) = \sin x + \cos^2 x$

12. $(\sin x)(\cot x + \cos x \tan x) = \cos x + \sin^2 x$

13. $(1 - \tan x)^2 = \sec^2 x - 2 \tan x$

14. $(\cos x - \sin x)^2 = 1 - 2 \sin x \cos x$

15. $\dfrac{(1 - \cos u)(1 + \cos u)}{\cos^2 u} = \tan^2 u$

16. $\tan x + \sec x = \dfrac{\cos x}{1 - \sin x}$

17. $\dfrac{\cos^2 x - 1}{\cos x} = -\tan x \sin x$

18. $\dfrac{\sec^2 \theta - 1}{\sin \theta} = \dfrac{\sin \theta}{1 - \sin^2 \theta}$

19. $(1 - \sin \beta)(1 + \csc \beta) = 1 - \sin \beta + \csc \beta - \sin \beta \csc \beta$

20. $\dfrac{1}{1 - \cos x} + \dfrac{1}{1 + \cos x} = 2 \csc^2 x$

21. $(\cos t - \sin t)^2 + (\cos t + \sin t)^2 = 2$

22. $\sin^2 \alpha - \cos^2 \alpha = 1 - 2 \cos^2 \alpha$

23. $\dfrac{1 + \tan^2 x}{\sin^2 x + \cos^2 x} = \sec^2 x$

24. $\dfrac{1}{\tan \beta} + \tan \beta = \sec \beta \csc \beta$

25. $\dfrac{\cos \beta}{1 + \sin \beta} = \dfrac{1 - \sin \beta}{\cos \beta}$

26. $\dfrac{\sec x + 1}{\tan x} = \dfrac{\sin x}{1 - \cos x}$

27. $\dfrac{\tan^2 x}{\sec x + 1} = \dfrac{1 - \cos x}{\cos x}$

28. $\dfrac{\cot v - 1}{\cot v + 1} = \dfrac{1 - \tan v}{1 + \tan v}$

29. $\cot^2 x - \cos^2 x = \cos^2 x \cot^2 x$

30. $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$

31. $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x$

32. $\tan^4 t + \tan^2 t = \sec^4 t - \sec^2 t$

33. $(x \sin \alpha + y \cos \alpha)^2 + (x \cos \alpha - y \sin \alpha)^2 = x^2 + y^2$

34. $\dfrac{1 - \cos \theta}{\sin \theta} = \dfrac{\sin \theta}{1 + \cos \theta}$

35. $\dfrac{\tan x}{\sec x - 1} = \dfrac{\sec x + 1}{\tan x}$

36. $\dfrac{\sin t}{1 + \cos t} + \dfrac{1 + \cos t}{\sin t} = 2 \csc t$

37. $\dfrac{\sin x - \cos x}{\sin x + \cos x} = \dfrac{2 \sin^2 x - 1}{1 + 2 \sin x \cos x}$

38. $\dfrac{1 + \cos x}{1 - \cos x} = \dfrac{\sec x + 1}{\sec x - 1}$

39. $\dfrac{\sin t}{1 - \cos t} + \dfrac{1 + \cos t}{\sin t} = \dfrac{2(1 + \cos t)}{\sin t}$

40. $\dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

41. $\sin^2 x \cos^3 x = (\sin^2 x - \sin^4 x)(\cos x)$

42. $\sin^5 x \cos^2 x = (\cos^2 x - 2\cos^4 x + \cos^6 x)(\sin x)$

43. $\cos^5 x = (1 - 2\sin^2 x + \sin^4 x)(\cos x)$

44. $\sin^3 x \cos^3 x = (\sin^3 x - \sin^5 x)(\cos x)$

45. $\dfrac{\tan x}{1 - \cot x} + \dfrac{\cot x}{1 - \tan x} = 1 + \sec x \csc x$

46. $\dfrac{\cos x}{1 + \sin x} + \dfrac{\cos x}{1 - \sin x} = 2 \sec x$

47. $\dfrac{2\tan x}{1 - \tan^2 x} + \dfrac{1}{2\cos^2 x - 1} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

48. $\dfrac{1 - 3\cos x - 4\cos^2 x}{\sin^2 x} = \dfrac{1 - 4\cos x}{1 - \cos x}$

49. $\cos^3 x = (1 - \sin^2 x)(\cos x)$

50. $\sec^4 x = (1 + \tan^2 x)(\sec^2 x)$

51. $\sin^5 x = (1 - 2\cos^2 x + \cos^4 x)(\sin x)$

In Exercises 52–57, match the function with an equivalent expression from the following list. Then confirm the match with a proof. (The matching is not one-to-one.)

(a) $\sec^2 x \csc^2 x$ **(b)** $\sec x + \tan x$ **(c)** $2\sec^2 x$

(d) $\tan x \sin x$ **(e)** $\sin x \cos x$

52. $\dfrac{1 + \sin x}{\cos x}$

53. $(1 + \sec x)(1 - \cos x)$

54. $\sec^2 x + \csc^2 x$

55. $\dfrac{1}{1 + \sin x} + \dfrac{1}{1 - \sin x}$

56. $\dfrac{1}{\tan x + \cot x}$

57. $\dfrac{1}{\sec x - \tan x}$

Standardized Test Questions

58. True or False The equation $\sqrt{x^2} = x$ is an identity. Justify your answer.

59. True or False The equation $(\sqrt{x})^2 = x$ is an identity. Justify your answer.

You should answer these questions without using a calculator.

60. Multiple Choice If $f(x) = g(x)$ is an identity with domain of validity D, which of the following must be true?

I. For any x in D, $f(x)$ is defined.

II. For any x in D, $g(x)$is defined.

III. For any x in D, $f(x) = g(x)$.

(A) None

(B) I and II only

(C) I and III only

(D) III only

(E) I, II, and III

61. Multiple Choice Which of these is an efficient first step in proving the identity $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$?

(A) $\dfrac{\sin x}{1 - \cos x} = \dfrac{\cos\left(\dfrac{\pi}{2} - x\right)}{1 - \cos x}$

(B) $\dfrac{\sin x}{1 - \cos x} = \dfrac{\sin x}{\sin^2 x + \cos^2 x - \cos x}$

(C) $\dfrac{\sin x}{1 - \cos x} = \dfrac{\sin x}{1 - \cos x} \cdot \dfrac{\csc x}{\csc x}$

(D) $\dfrac{\sin x}{1 - \cos x} = \dfrac{\sin x}{1 - \cos x} \cdot \dfrac{1 - \cos x}{1 - \cos x}$

(E) $\dfrac{\sin x}{1 - \cos x} = \dfrac{\sin x}{1 - \cos x} \cdot \dfrac{1 + \cos x}{1 + \cos x}$

62. Multiple Choice Which of the following could be an intermediate expression in a proof of the identity

$\tan\theta + \sec\theta = \dfrac{\cos\theta}{1 - \sin\theta}$?

(A) $\sin\theta + \cos\theta$

(B) $\tan\theta + \csc\theta$

(C) $\dfrac{\sin\theta + 1}{\cos\theta}$

(D) $\dfrac{\cos\theta}{1 + \sin\theta}$

(E) $\cos\theta - \cot\theta$

63. Multiple Choice If $f(x) = g(x)$ is an identity and $\dfrac{f(x)}{g(x)} = k$, which of the following must be *false*?

(A) $g(x) \neq 0$

(B) $f(x) = 0$

(C) $k = 1$

(D) $f(x) - g(x) = 0$

(E) $f(x)g(x) > 0$

Explorations

In Exercises 64–69, identify a simple function that has the same graph. Then confirm your choice with a proof.

64. $\sin x \cot x$

65. $\cos x \tan x$

66. $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x}$

67. $\dfrac{\csc x}{\sin x} - \dfrac{\cot x \csc x}{\sec x}$

68. $\dfrac{\sin x}{\tan x}$

69. $(\sec^2 x)(1 - \sin^2 x)$

70. Writing to Learn Let θ be any number that is in the domain of all six trig functions. Explain why the natural logarithms of all six basic trig functions of θ sum to 0.

71. If A and B are complementary angles, prove that $\sin^2 A + \sin^2 B = 1$.

72. Group Activity If your class contains $2n$ students, write the two expressions from n different identities on separate pieces of paper. (If your class contains an odd number of students, invite your teacher to join you for this activity.) You can use the identities from Exercises 11–51 in this section or from other textbooks, but be sure to write them all in the variable x. Mix up the slips of paper and give one to each student in your class. Then see how long it takes you as a class, without looking at the book, to pair yourselves off as identities. (This activity takes on an added degree of difficulty if you try it without calculators.)

Extending the Ideas

In Exercises 73–78, confirm the identity.

73. $\sqrt{\dfrac{1-\sin t}{1+\sin t}} = \dfrac{1-\sin t}{|\cos t|}$

74. $\sqrt{\dfrac{1+\cos t}{1-\cos t}} = \dfrac{1+\cos t}{|\sin t|}$

75. $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x$

76. $\cos^6 x - \sin^6 x = (\cos^2 x - \sin^2 x)(1 - \cos^2 x \sin^2 x)$

77. $\ln|\tan x| = \ln|\sin x| - \ln|\cos x|$

78. $\ln|\sec\theta + \tan\theta| + \ln|\sec\theta - \tan\theta| = 0$

79. Writing to Learn Let $y_1 = [\sin(x + 0.001) - \sin x]/0.001$ and $y_2 = \cos x$.

(a) Use graphs and tables to decide whether $y_1 = y_2$.

(b) Find a value for h so that the graph of $y_3 = y_1 - y_2$ in $[-2\pi, 2\pi]$ by $[-h, h]$ appears to be a sinusoid. Give a convincing argument that y_3 is a sinusoid.

80. Hyperbolic Functions The *hyperbolic trigonometric functions* are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

Confirm the identity.

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(c) $\coth^2 x - 1 = \operatorname{csch}^2 x$

81. Writing to Learn Write a paragraph to explain why

$$\cos x = \cos x + \sin(10\pi x)$$

appears to be an identity when the two sides are graphed in a decimal window. Give a convincing argument that it is not an identity.