4.7 Inverse Trigonometric Functions

What you'll learn about

- Inverse Sine Function
- Inverse Cosine and Tangent Functions
- Composing Trigonometric and Inverse Trigonometric Functions
- Applications of Inverse Trigonometric Functions

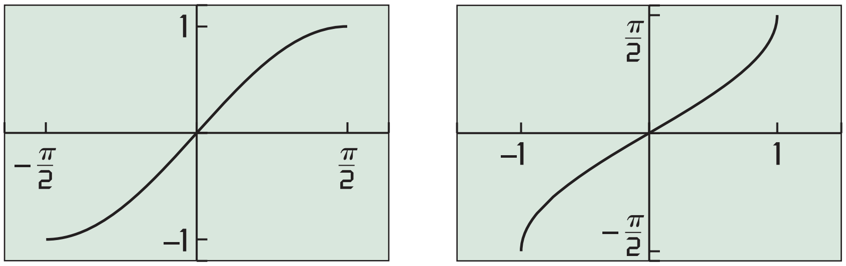
... and why

Inverse trig functions can be used to solve trigonometric equations.

Inverse Sine Function

You learned in Section 1.4 that each function has an inverse relation, and that this inverse relation is a function only if the original function is one-to-one. The six basic trigonometric functions, being periodic, fail the horizontal line test for one-to-oneness rather spectacularly. However, you also learned in Section 1.4 that some functions are important enough that we want to study their inverse behavior despite the fact that they are not one-to-one. We do this by restricting the domain of the original function to an interval on which it *is* one-to-one, then finding the inverse of the restricted function. (We did this when defining the square root function, which is the inverse of the function $y = x^2$ restricted to a nonnegative domain.)

If you restrict the domain of $y = \sin x$ to the interval $[-\pi/2, \pi/2]$, as shown in Figure 4.69a, the restricted function is one-to-one. The **inverse sine function** $y = \sin^{-1} x$ is the inverse of this restricted portion of the sine function (Figure 4.69b).



[–2, 2] by [–1.2, 1.2]
(a)

[–1.5, 1.5] by [–1.7, 1.7]
(b)

FIGURE 4.69 The (a) restriction of $y = \sin x$ is one-to-one and (b) has an inverse, $y = \sin^{-1} x$.

By the usual inverse relationship, the statements

$$y = \sin^{-1} x \quad \text{and} \quad x = \sin y$$

are equivalent for y-values in the restricted domain $[-\pi/2, \pi/2]$ and x-values in $[-1, 1]$. This means that $\sin^{-1} x$ can be thought of as *the angle between $-\pi/2$ and $\pi/2$ whose sine is x*. Since angles and directed arcs on the unit circle have the same measure, the angle $\sin^{-1} x$ is also called the **arcsine of x**.

Inverse Sine Function (Arcsine Function)

The unique angle y in the interval $[-\pi/2, \pi/2]$ such that $\sin y = x$ is the **inverse sine** (or **arcsine**) of x, denoted $\sin^{-1} x$ or **arcsin x**.

The domain of $y = \sin^{-1} x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.

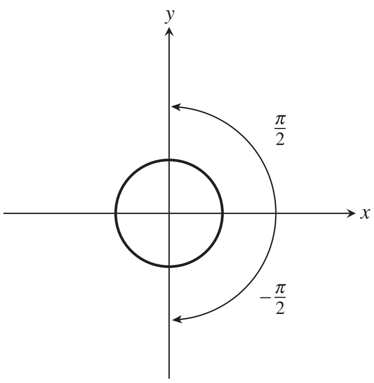


FIGURE 4.70 The values of $y = \sin^{-1} x$ will always be found on the right-hand side of the unit circle, between $-\pi/2$ and $\pi/2$.

It helps to think of the range of $y = \sin^{-1} x$ as being along the right-hand side of the unit circle, which is traced out as angles range from $-\pi/2$ to $\pi/2$ (Figure 4.70).

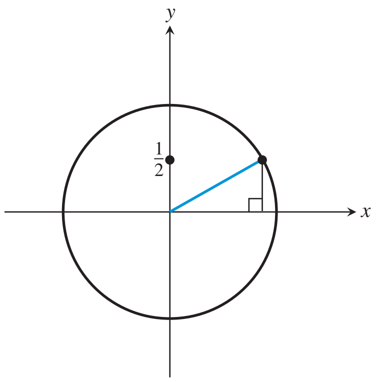


FIGURE 4.71 $\sin^{-1}(1/2) = \pi/6$. (Example 1a)

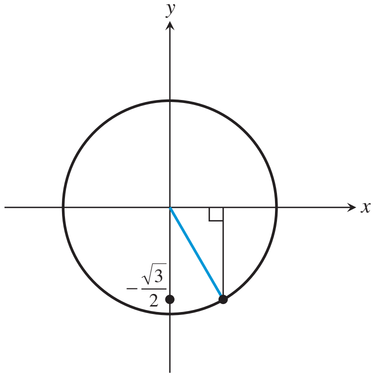


FIGURE 4.72 $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$. (Example 1b)

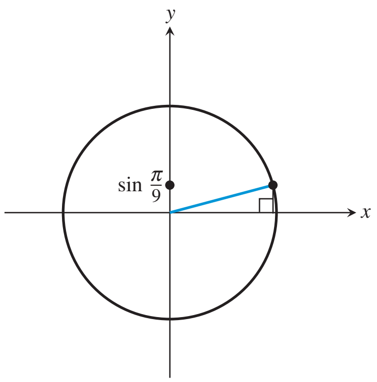


FIGURE 4.73 $\sin^{-1}(\sin(\pi/9)) = \pi/9$. (Example 1d)

EXAMPLE 1 Evaluating $\sin^{-1} x$ without a Calculator

Find the exact value of each expression without a calculator.

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ **(b)** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **(c)** $\sin^{-1}\left(\frac{\pi}{2}\right)$

(d) $\sin^{-1}\left(\sin\left(\frac{\pi}{9}\right)\right)$ **(e)** $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

SOLUTION

(a) Find the point on the right half of the unit circle whose y-coordinate is $1/2$ and draw a reference triangle (Figure 4.71). We recognize this as one of our special ratios, and the angle in the interval $[-\pi/2, \pi/2]$ whose sine is $1/2$ is $\pi/6$. Therefore

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

(b) Find the point on the right half of the unit circle whose y-coordinate is $-\sqrt{3}/2$ and draw a reference triangle (Figure 4.72). We recognize this as one of our special ratios, and the angle in the interval $[-\pi/2, \pi/2]$ whose sine is $-\sqrt{3}/2$ is $-\pi/3$. Therefore

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}.$$

(c) $\sin^{-1}(\pi/2)$ does not exist, because the domain of $\sin^{-1}$ is $[-1, 1]$ and $\pi/2 > 1$.

(d) Draw an angle of $\pi/9$ in standard position and mark its y-coordinate on the y-axis (Figure 4.73). The angle in the interval $[-\pi/2, \pi/2]$ whose sine is this number is $\pi/9$. Therefore

$$\sin^{-1}\left(\sin\left(\frac{\pi}{9}\right)\right) = \frac{\pi}{9}.$$

(e) Draw an angle of $5\pi/6$ in standard position (notice that this angle is *not* in the interval $[-\pi/2, \pi/2]$) and mark its y-coordinate on the y-axis. (See Figure 4.74 on the next page.) The angle in the interval $[-\pi/2, \pi/2]$ whose sine is this number is $\pi - 5\pi/6 = \pi/6$. Therefore

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \frac{\pi}{6}.$$

Now try Exercise 1.

EXAMPLE 2 Evaluating $\sin^{-1} x$ with a Calculator

Use a calculator in radian mode to evaluate these inverse sine values:

(a) $\sin^{-1}(-0.81)$ **(b)** $\sin^{-1}(\sin(3.49\pi))$

SOLUTION

(a) $\sin^{-1}(-0.81) = -0.9441521\ldots \approx -0.944$

(b) $\sin^{-1}(\sin(3.49\pi)) = -1.5393804\ldots \approx -1.539$

Although this is a calculator answer, we can use it to get an exact answer if we are alert enough to expect a multiple of π. Divide the answer by π:

$$\text{Ans}/\pi = -0.49$$

Therefore, we conclude that $\sin^{-1}(\sin(3.49\pi)) = -0.49\pi$.

You should also try to work Example 2b without a calculator. It is possible!

Now try Exercise 19.

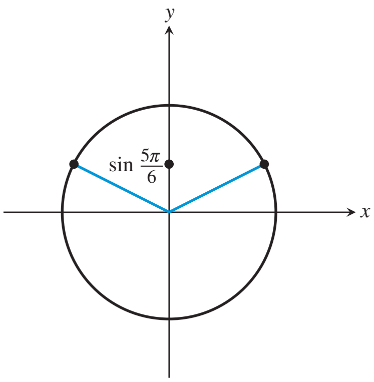


FIGURE 4.74 $\sin^{-1}(\sin(5\pi/6)) = \pi/6$. (Example 1e)

What about the Inverse Composition Rule?

Does Example 1e violate the Inverse Composition Rule of Section 1.4? That rule guarantees that $f^{-1}(f(x)) = x$ for every x in the domain of f. Keep in mind, however, that the domain of f might need to be restricted in order for f^{-1} to exist. That is certainly the case with the sine function. So Example 1e does not violate the Inverse Composition Rule, because that rule *does not apply* at $x = 5\pi/6$. It lies outside the (restricted) domain of sine.

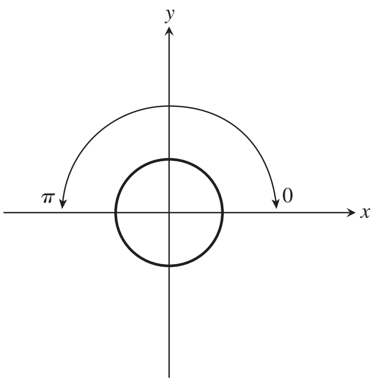
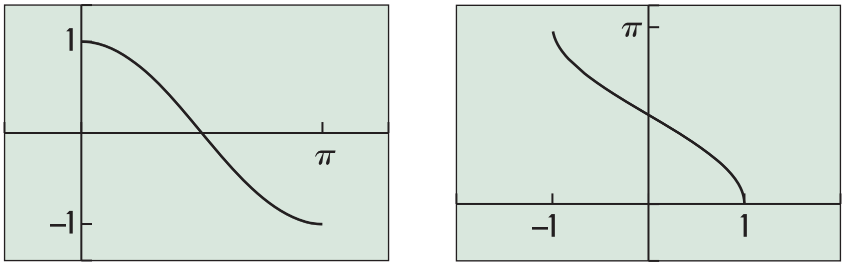


FIGURE 4.76 The values of $y = \cos^{-1} x$ will always be found on the top half of the unit circle, between 0 and π.

Inverse Cosine and Tangent Functions

If you restrict the domain of $y = \cos x$ to the interval $[0, \pi]$, as shown in Figure 4.75a, the restricted function is one-to-one. The **inverse cosine function** $y = \cos^{-1} x$ is the inverse of this restricted portion of the cosine function (Figure 4.75b).



[–1, 4] by [–1.4, 1.4]
(a)

[–2, 2] by [–1, 3.5]
(b)

FIGURE 4.75 The (a) restriction of $y = \cos x$ is one-to-one and (b) has an inverse, $y = \cos^{-1} x$.

By the usual inverse relationship, the statements

$$y = \cos^{-1} x \quad \text{and} \quad x = \cos y$$

are equivalent for y-values in the restricted domain $[0, \pi]$ and x-values in $[-1, 1]$. This means that $\cos^{-1} x$ can be thought of as *the angle between 0 and π whose cosine is x.* The angle $\cos^{-1} x$ is also the **arccosine of x**.

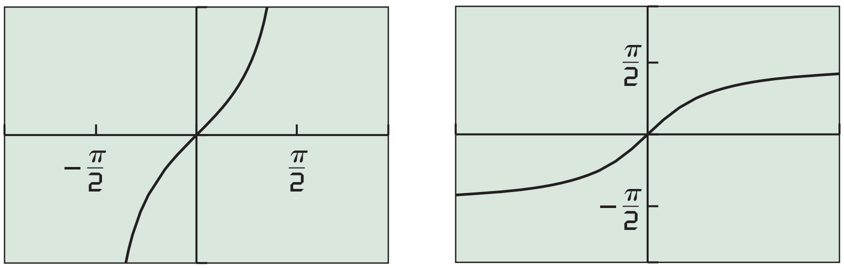
Inverse Cosine Function (Arccosine Function)

The unique angle y in the interval $[0, \pi]$ such that $\cos y = x$ is the **inverse cosine** (or **arccosine**) of x, denoted $\cos^{-1} x$ or **arccos x**.

The domain of $y = \cos^{-1} x$ is $[-1, 1]$ and the range is $[0, \pi]$.

It helps to think of the range of $y = \cos^{-1} x$ as being along the top half of the unit circle, which is traced out as angles range from 0 to π (Figure 4.76).

If you restrict the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$, as shown in Figure 4.77a, the restricted function is one-to-one. The **inverse tangent function** $y = \tan^{-1} x$ is the inverse of this restricted portion of the tangent function (Figure 4.77b).



[–3, 3] by [–2, 2]
(a)

[–4, 4] by [–2.8, 2.8]
(b)

FIGURE 4.77 The (a) restriction of $y = \tan x$ is one-to-one and (b) has an inverse, $y = \tan^{-1} x$.

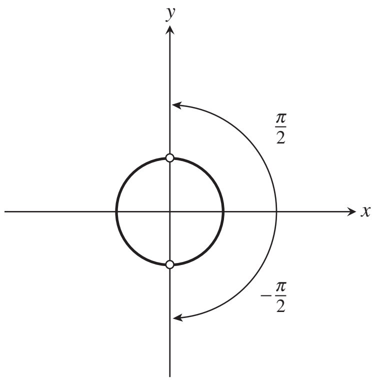


FIGURE 4.78 The values of $y = \tan^{-1} x$ will always be found on the right-hand side of the unit circle, between (but not including) $-\pi/2$ and $\pi/2$.

By the usual inverse relationship, the statements

$$y = \tan^{-1} x \quad \text{and} \quad x = \tan y$$

are equivalent for y-values in the restricted domain $(-\pi/2, \pi/2)$ and x-values in $(-\infty, \infty)$. This means that $\tan^{-1} x$ can be thought of as *the angle between* $-\pi/2$ *and* $\pi/2$ *whose tangent is* x. The angle $\tan^{-1} x$ is also the **arctangent of x**.

Inverse Tangent Function (Arctangent Function)

The unique angle y in the interval $(-\pi/2, \pi/2)$ such that $\tan y = x$ is the **inverse tangent** (or **arctangent**) of x, denoted $\tan^{-1} x$ or **arctan x**.
The domain of $y = \tan^{-1} x$ is $(-\infty, \infty)$ and the range is $(-\pi/2, \pi/2)$.

It helps to think of the range of $y = \tan^{-1} x$ as being along the right-hand side of the unit circle (minus the top and bottom points), which is traced out as angles range from $-\pi/2$ to $\pi/2$ (noninclusive) (Figure 4.78).

EXAMPLE 3 Evaluating Inverse Trig Functions without a Calculator

Find the exact value of the expression without a calculator.

(a) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

(b) $\tan^{-1} \sqrt{3}$

(c) $\cos^{-1}(\cos(-1.1))$

SOLUTION

(a) Find the point on the top half of the unit circle whose x-coordinate is $-\sqrt{2}/2$ and draw a reference triangle (Figure 4.79). We recognize this as one of our special ratios, and the angle in the interval $[0, \pi]$ whose cosine is $-\sqrt{2}/2$ is $3\pi/4$. Therefore

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}.$$

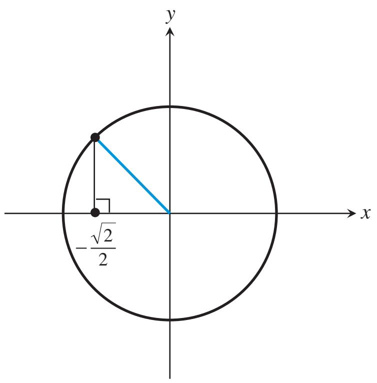


FIGURE 4.79 $\cos^{-1}(-\sqrt{2}/2) = 3\pi/4$. (Example 3a)

(b) Find the point on the right side of the unit circle whose y-coordinate is $\sqrt{3}$ times its x-coordinate and draw a reference triangle (Figure 4.80). We recognize this as one of our special ratios, and the angle in the interval $(-\pi/2, \pi/2)$ whose tangent is $\sqrt{3}$ is $\pi/3$. Therefore

$$\tan^{-1} \sqrt{3} = \frac{\pi}{3}.$$

(c) Draw an angle of -1.1 in standard position (notice that this angle is *not* in the interval $[0, \pi]$) and mark its x-coordinate on the x-axis (Figure 4.81). The angle in the interval $[0, \pi]$ whose cosine is this number is 1.1. Therefore

$$\cos^{-1}(\cos(-1.1)) = 1.1.$$

Now try Exercises 5 and 7.

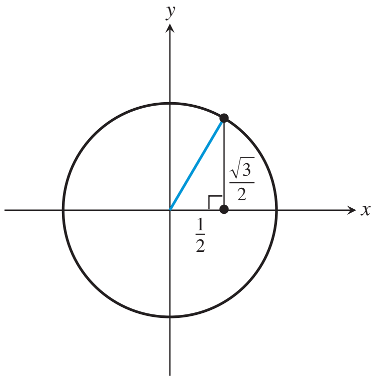


FIGURE 4.80 $\tan^{-1}\sqrt{3} = \pi/3$. (Example 3b)

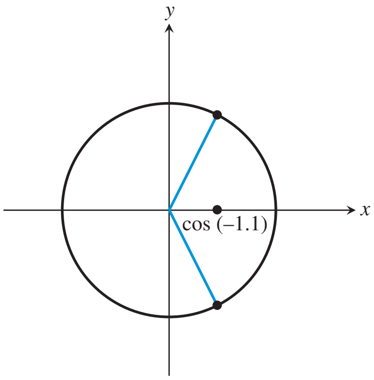


FIGURE 4.81 $\cos^{-1}(\cos(-1.1)) = 1.1$. (Example 3c)

EXAMPLE 4 Describing End Behavior

Describe the end behavior of the function $y = \tan^{-1} x$.

SOLUTION We can get this information most easily by considering the graph of $y = \tan^{-1} x$, remembering how it relates to the restricted graph of $y = \tan x$. (See Figure 4.82.)

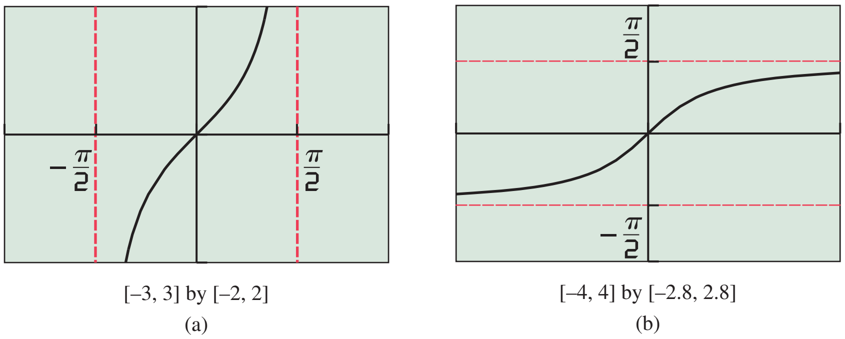


FIGURE 4.82 The graphs of (a) $y = \tan x$ (restricted) and (b) $y = \tan^{-1} x$. The vertical asymptotes of $y = \tan x$ are reflected to become the horizontal asymptotes of $y = \tan^{-1} x$. (Example 4)

When we reflect the graph of $y = \tan x$ about the line $y = x$ to get the graph of $y = \tan^{-1} x$, the vertical asymptotes $x = \pm\pi/2$ become horizontal asymptotes $y = \pm\pi/2$. We can state the end behavior accordingly:

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x\to+\infty} \tan^{-1} x = \frac{\pi}{2}.$$

Now try Exercise 21.

What about Arccot, Arcsec, and Arccsc?

Because we already have inverse functions for their reciprocals, we do not really need inverse functions for cot, sec, and csc for computational purposes. Moreover, the decision of how to choose the range of arcsec and arccsc is not as straightforward as with the other functions. See Exercises 63, 71, and 72.

Composing Trigonometric and Inverse Trigonometric Functions

We have already seen the need for caution when applying the Inverse Composition Rule to the trigonometric functions and their inverses (Examples 1e and 3c). The following equations are *always* true whenever they are defined:

$$\sin(\sin^{-1}(x)) = x \qquad \cos(\cos^{-1}(x)) = x \qquad \tan(\tan^{-1}(x)) = x$$

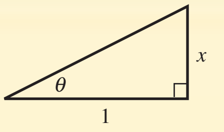
On the other hand, the following equations are true only for x-values in the "restricted" domains of sin, cos, and tan:

$$\sin^{-1}(\sin(x)) = x \quad \cos^{-1}(\cos(x)) = x \quad \tan^{-1}(\tan(x)) = x$$

An even more interesting phenomenon occurs when we compose inverse trigonometric functions of one kind with trigonometric functions of another kind, as in $\sin(\tan^{-1}x)$. Surprisingly, these trigonometric compositions reduce to algebraic functions that involve no trigonometry at all! This curious situation has profound implications in calculus, where it is sometimes useful to decompose nontrigonometric functions into trigonometric components that seem to come out of nowhere. Try Exploration 1.

EXPLORATION 1 Finding Inverse Trig Functions of Trig Functions

In the right triangle shown to the right, the angle θ is measured in radians.



1. Find $\tan\theta$.
2. Find $\tan^{-1} x$.
3. Find the hypotenuse of the triangle as a function of x.
4. Find $\sin(\tan^{-1}(x))$ as a ratio involving no trig functions.
5. Find $\sec(\tan^{-1}(x))$ as a ratio involving no trig functions.
6. If $x < 0$, then $\tan^{-1} x$ is a negative angle in the fourth quadrant (Figure 4.83). Verify that your answers to parts (4) and (5) are still valid in this case.

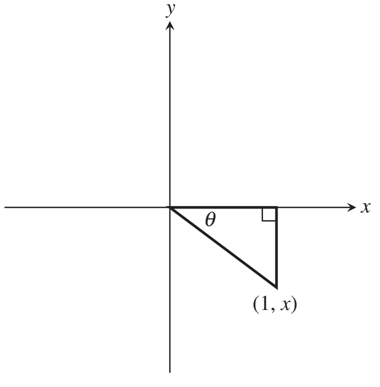


FIGURE 4.83 If $x < 0$, then $\theta = \tan^{-1} x$ is an angle in the fourth quadrant. (Exploration 1)

EXAMPLE 5 Composing Trig Functions with Arcsine

Compose each of the six basic trig functions with $\sin^{-1} x$ and reduce the composite function to an algebraic expression involving no trig functions.

SOLUTION This time we begin with the triangle shown in Figure 4.84, in which $\theta = \sin^{-1} x$. (This triangle could appear in the fourth quadrant if x were negative, but the trig ratios would be the same.)

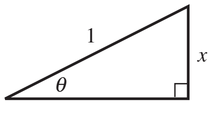


FIGURE 4.84 A triangle in which $\theta = \sin^{-1} x$. (Example 5)

The remaining side of the triangle (which is $\cos\theta$) can be found by the Pythagorean Theorem. If we denote the unknown side by s, we have

$$\begin{aligned} s^2 + x^2 &= 1 \\ s^2 &= 1 - x^2 \\ s &= \pm\sqrt{1 - x^2} \end{aligned}$$

Note the ambiguous sign, which requires a further look. Since $\sin^{-1} x$ is always in Quadrant I or IV, the horizontal side of the triangle can only be positive.

Therefore, we can actually write s unambiguously as $\sqrt{1 - x^2}$, giving us the triangle in Figure 4.85.

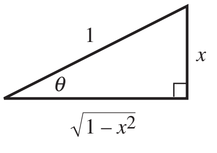


FIGURE 4.85 If $\theta = \sin^{-1} x$, then $\cos\theta = \sqrt{1 - x^2}$. Note that $\cos\theta$ will be positive because $\sin^{-1} x$ can only be in Quadrant I or IV. (Example 5)

(continued)

We can now read all the required ratios straight from the triangle:

$$\sin(\sin^{-1}(x)) = x \qquad \csc(\sin^{-1}(x)) = \frac{1}{x}$$

$$\cos(\sin^{-1}(x)) = \sqrt{1 - x^2} \qquad \sec(\sin^{-1}(x)) = \frac{1}{\sqrt{1 - x^2}}$$

$$\tan(\sin^{-1}(x)) = \frac{x}{\sqrt{1 - x^2}} \qquad \cot(\sin^{-1}(x)) = \frac{\sqrt{1 - x^2}}{x}$$

Now try Exercise 47.

Applications of Inverse Trigonometric Functions

When an application involves an angle as a dependent variable, as in $\theta = f(x)$, then to solve for x, it is natural to use an inverse trigonometric function and find $x = f^{-1}(\theta)$.

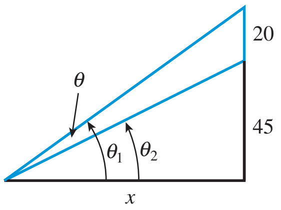


FIGURE 4.86 The diagram for the stadium screen. (Example 6)

EXAMPLE 6 Calculating a Viewing Angle

The bottom of a 20-foot replay screen at Dodger Stadium is 45 feet above the playing field. As you move away from the wall, the angle formed by the screen at your eye changes. There is a distance from the wall at which the angle is the greatest. What is that distance?

SOLUTION

Model

The angle subtended by the screen is represented in Figure 4.86 by θ, and $\theta = \theta_1 - \theta_2$. Since $\tan\theta_1 = 65/x$, it follows that $\theta_1 = \tan^{-1}(65/x)$. Similarly, $\theta_2 = \tan^{-1}(45/x)$. Thus,

$$\theta = \tan^{-1}\frac{65}{x} - \tan^{-1}\frac{45}{x}.$$

Solve Graphically

Figure 4.87 shows a graph of θ that reflects degree mode. The question about distance for maximum viewing angle can be answered by finding the x-coordinate of the maximum point of this graph. Using grapher methods we see that this maximum occurs when $x \approx 54$ feet.

Therefore the maximum angle subtended by the replay screen occurs about 54 feet from the wall.

Now try Exercise 55.

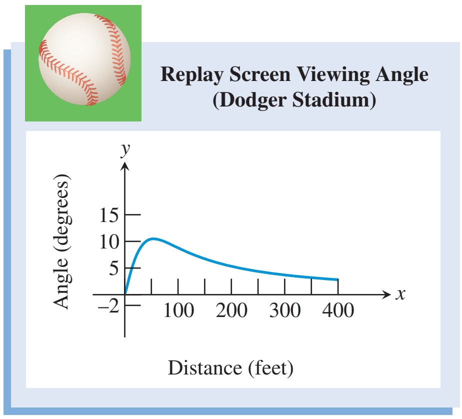


FIGURE 4.87 Viewing angle θ as a function of distance x from the wall. (Example 6)

QUICK REVIEW 4.7 *(For help, go to Section 4.3.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, state the sign (positive or negative) of the sine, cosine, and tangent in the quadrant.

1. Quadrant I
2. Quadrant II
3. Quadrant III
4. Quadrant IV

In Exercises 5–10, find the exact value.

5. $\sin(\pi/6)$
6. $\tan(\pi/4)$
7. $\cos(2\pi/3)$
8. $\sin(2\pi/3)$
9. $\sin(-\pi/6)$
10. $\cos(-\pi/3)$

SECTION 4.7 EXERCISES

In Exercises 1–12, find the exact value.

1. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **2.** $\sin^{-1}\left(-\frac{1}{2}\right)$

3. $\tan^{-1} 0$ **4.** $\cos^{-1} 1$

5. $\cos^{-1}\left(\frac{1}{2}\right)$ **6.** $\tan^{-1} 1$

7. $\tan^{-1}(-1)$ **8.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

9. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **10.** $\tan^{-1}(-\sqrt{3})$

11. $\cos^{-1} 0$ **12.** $\sin^{-1} 1$

In Exercises 13–16, use a calculator to find the approximate value. Express your answer in degrees.

13. $\sin^{-1}(0.362)$ **14.** arcsin 0.67

15. $\tan^{-1}(-12.5)$ **16.** $\cos^{-1}(-0.23)$

In Exercises 17–20, use a calculator to find the approximate value. Express your result in radians.

17. $\tan^{-1}(2.37)$ **18.** $\tan^{-1}(22.8)$

19. $\sin^{-1}(-0.46)$ **20.** $\cos^{-1}(-0.853)$

In Exercises 21 and 22, describe the end behavior of the function.

21. $y = \tan^{-1}(x^2)$ **22.** $y = (\tan^{-1} x)^2$

In Exercises 23–32, find the exact value without a calculator.

23. $\cos(\sin^{-1}(1/2))$ **24.** $\sin(\tan^{-1} 1)$

25. $\sin^{-1}(\cos(\pi/4))$ **26.** $\cos^{-1}(\cos(7\pi/4))$

27. $\cos(2\sin^{-1}(1/2))$ **28.** $\sin(\tan^{-1}(-1))$

29. $\arcsin(\cos(\pi/3))$ **30.** $\arccos(\tan(\pi/4))$

31. $\cos(\tan^{-1}\sqrt{3})$ **32.** $\tan^{-1}(\cos\pi)$

In Exercises 33–36, analyze each function for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

33. $f(x) = \sin^{-1} x$

34. $f(x) = \cos^{-1} x$

35. $f(x) = \tan^{-1} x$

36. $f(x) = \cot^{-1} x$ (See graph in Exercise 67.)

In Exercises 37–40, use transformations to describe how the graph of the function is related to a basic inverse trigonometric graph. State the domain and range.

37. $f(x) = \sin^{-1}(2x)$ **38.** $g(x) = 3\cos^{-1}(2x)$

39. $h(x) = 5\tan^{-1}(x/2)$ **40.** $g(x) = 3\arccos(x/2)$

In Exercises 41–46, find the solution to the equation without a calculator.

41. $\sin(\sin^{-1} x) = 1$ **42.** $\cos^{-1}(\cos x) = 1$

43. $2\sin x = 1$ **44.** $\tan x = -1$

45. $\cos(\cos^{-1} x) = 1/3$ **46.** $\sin^{-1}(\sin x) = \pi/10$

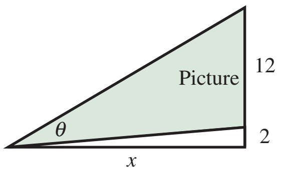
In Exercises 47–52, find an algebraic expression equivalent to the given expression. (*Hint:* Form a right triangle as done in Example 5.)

47. $\sin(\tan^{-1} x)$ **48.** $\cos(\tan^{-1} x)$

49. $\tan(\arcsin x)$ **50.** $\cot(\arccos x)$

51. $\cos(\arctan 2x)$ **52.** $\sin(\arccos 3x)$

53. Group Activity Viewing Angle You are standing in an art museum viewing a picture. The bottom of the picture is 2 ft above your eye level, and the picture is 12 ft tall. Angle θ is formed by the lines of vision to the bottom and to the top of the picture.

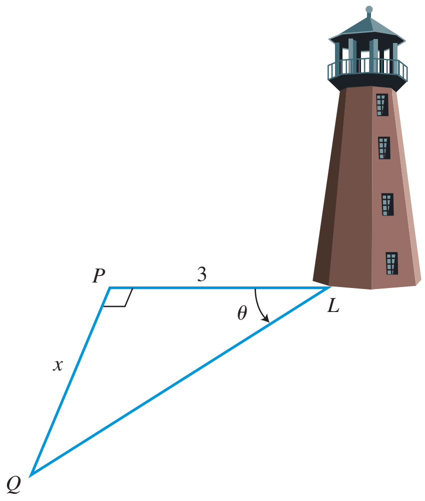


(a) Show that $\theta = \tan^{-1}\left(\frac{14}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)$.

(b) Graph θ in the $[0, 25]$ by $[0, 55]$ viewing window using degree mode. Use your grapher to show that the maximum value of θ occurs approximately 5.3 ft from the picture.

(c) How far (to the nearest foot) are you standing from the wall if $\theta = 35°$?

54. Group Activity Analysis of a Lighthouse
A rotating beacon L stands 3 m across the harbor from the nearest point P along a straight shoreline. As the light rotates, it forms an angle θ as shown in the figure, and illuminates a point Q on the same shoreline as P.

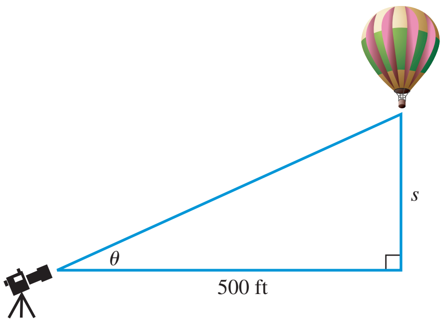


(a) Show that $\theta = \tan^{-1}\left(\frac{x}{3}\right)$.

(b) Graph θ in the viewing window $[-20, 20]$ by $[-90, 90]$ using degree mode. What do negative values of x represent in the problem? What does a positive angle represent? A negative angle?

(c) Find θ when $x = 15$.

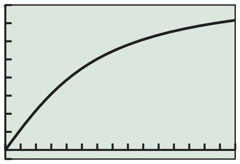
55. Rising Hot-Air Balloon The hot-air balloon festival held each year in Phoenix, Arizona, is a popular event for photographers. Jo Silver, an award-winning photographer at the event, watches a balloon rising from ground level from a point 500 ft away on level ground.



(a) Write θ as a function of the height s of the balloon.

(b) Is the change in θ greater as s changes from 10 ft to 20 ft, or as s changes from 200 ft to 210 ft? Explain.

(c) **Writing to Learn** In the graph of this relationship shown here, do you think that the x-axis represents the height s and the y-axis angle θ, or does the x-axis represent angle θ and the y-axis height s? Explain.



[0, 1500] by [–5, 80]

56. Find the domain and range of each of the following functions.

(a) $f(x) = \sin(\sin^{-1} x)$

(b) $g(x) = \sin^{-1}(x) + \cos^{-1}(x)$

(c) $h(x) = \sin^{-1}(\sin x)$

(d) $k(x) = \sin(\cos^{-1} x)$

(e) $q(x) = \cos^{-1}(\sin x)$

Standardized Test Questions

57. True or False $\sin(\sin^{-1} x) = x$ for all real numbers x. Justify your answer.

58. True or False The graph of $y = \arctan x$ has two horizontal asymptotes. Justify your answer.

You should answer these questions without using a calculator.

59. Multiple Choice $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) =$

(A) $-\frac{7\pi}{6}$ (B) $-\frac{\pi}{3}$ (C) $-\frac{\pi}{6}$

(D) $\frac{2\pi}{3}$ (E) $\frac{5\pi}{6}$

60. Multiple Choice $\sin^{-1}(\sin \pi) =$

(A) -2π (B) $-\pi$ (C) 0

(D) π (E) 2π

61. Multiple Choice $\sec(\tan^{-1} x) =$

(A) x (B) $\csc x$ (C) $\sqrt{1 + x^2}$

(D) $\sqrt{1 - x^2}$ (E) $\frac{\sin x}{(\cos x)^2}$

62. Multiple Choice The range of the function $f(x) = \arcsin x$ is

(A) $(-\infty, \infty)$. (B) $(-1, 1)$. (C) $[-1, 1]$.

(D) $[0, \pi]$. (E) $[-\pi/2, \pi/2]$.

Explorations

63. Writing to Learn Using the format demonstrated in this section for the inverse sine, cosine, and tangent functions, give a careful definition of the inverse cotangent function. [*Hint:* The range of $y = \cot^{-1} x$ is $(0, \pi)$.]

64. Writing to Learn Use an appropriately labeled triangle to explain why $\sin^{-1} x + \cos^{-1} x = \pi/2$. For what values of x is the left-hand side of this equation defined?

65. Graph each of the following functions and interpret the graph to find the domain, range, and period of each function. Which of the three functions has points of discontinuity? Are the discontinuities removable or nonremovable?

(a) $y = \sin^{-1}(\sin x)$

(b) $y = \cos^{-1}(\cos x)$

(c) $y = \tan^{-1}(\tan x)$

Extending the Ideas

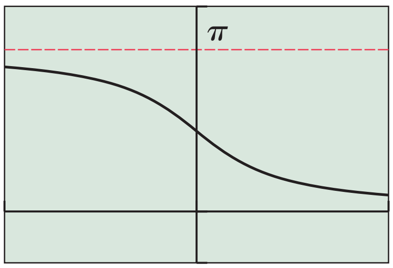
66. Practicing for Calculus Express each of the following functions as an algebraic expression involving no trig functions.

(a) $\cos(\sin^{-1} 2x)$ (b) $\sec^2(\tan^{-1} x)$

(c) $\sin(\cos^{-1} \sqrt{x})$ (d) $-\csc^2(\cot^{-1} x)$

(e) $\tan(\sec^{-1} x^2)$

67. Arccotangent on the Calculator Most graphing calculators do not have a button for the inverse cotangent. The graph is shown below. Find an expression that you can put into your calculator to produce a graph of $y = \cot^{-1} x$.



[–3, 3] by [–1, 4]

68. Advanced Decomposition Decompose each of the following algebraic functions by writing it as a trig function of an arctrig function.

(a) $\sqrt{1 - x^2}$ **(b)** $\dfrac{x}{\sqrt{1 + x^2}}$ **(c)** $\dfrac{x}{\sqrt{1 - x^2}}$

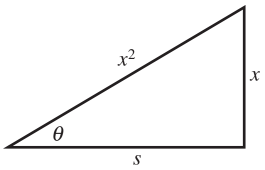
69. Use elementary transformations and the arctangent function to construct a function with domain all real numbers that has horizontal asymptotes at $y = 24$ and $y = 42$.

70. Avoiding Ambiguities When choosing the right triangle in Example 5, we used a hypotenuse of 1. It is sometimes necessary to use a variable quantity for the hypotenuse, in which case it is a good idea to use x^2 rather than x, just in case x is negative. (All of our definitions of the trig functions have involved triangles in which the hypotenuse is assumed to be positive.)

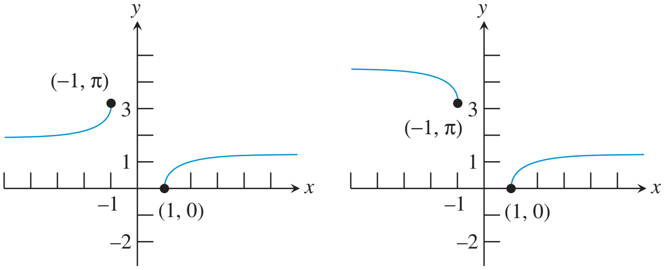
(a) If we use the triangle below to represent $\theta = \sin^{-1}(1/x)$, explain why side s must be positive regardless of the sign of x.

(b) Use the triangle in part (a) to find $\tan(\sin^{-1}(1/x))$.

(c) Using an appropriate triangle, find $\sin(\cos^{-1}(1/x))$.



71. Defining Arcsecant The range of the secant function is $(-\infty, -1] \cup [1, \infty)$, which must become the domain of the arcsecant function. The graph of $y = \text{arcsec } x$ must therefore be the union of two unbroken curves. Two possible graphs with the correct domain are shown below.



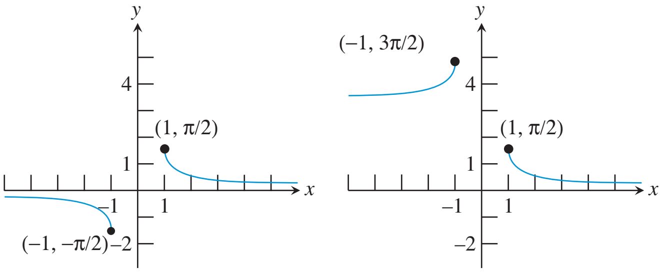
(a) The graph on the left has one horizontal asymptote. What is it?

(b) The graph on the right has two horizontal asymptotes. What are they?

(c) Which of these graphs is also the graph of $y = \cos^{-1}(1/x)$?

(d) Which of these graphs is increasing on both connected intervals?

72. Defining Arccosecant The range of the cosecant function is $(-\infty, -1] \cup [1, \infty)$, which must become the domain of the arccosecant function. The graph of $y = \text{arccsc } x$ must therefore be the union of two unbroken curves. Two possible graphs with the correct domain are shown below.



(a) The graph on the left has one horizontal asymptote. What is it?

(b) The graph on the right has two horizontal asymptotes. What are they?

(c) Which of these graphs is also the graph of $y = \sin^{-1}(1/x)$?

(d) Which of these graphs is decreasing on both connected intervals?