4.4 Graphs of Sine and Cosine: Sinusoids

What you'll learn about

- The Basic Waves Revisited
- Sinusoids and Transformations
- Modeling Periodic Behavior with Sinusoids

... and why

Sine and cosine gain added significance when used to model waves and periodic behavior

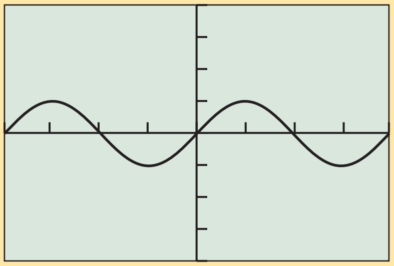
The Basic Waves Revisited

In the first three sections of this chapter you saw how the trigonometric functions are rooted in the geometry of triangles and circles. It is these connections with geometry that give trigonometric functions their mathematical power and make them widely applicable in many fields.

The unit circle in Section 4.3 was the key to defining the trigonometric functions as functions of real numbers. This makes them available for the same kind of analysis as the other functions introduced in Chapter 1. (Indeed, two of our "Twelve Basic Functions" are trigonometric.) We now take a closer look at the algebraic, graphical, and numerical properties of the trigonometric functions, beginning with sine and cosine.

Recall that we can learn quite a bit about the sine function by looking at its graph. Here is a summary of sine facts:

BASIC FUNCTION The Sine Function



$[-2\pi, 2\pi]$ by $[-4, 4]$

FIGURE 4.37A

$f(x) = \sin x$
Domain: All reals
Range: $[-1, 1]$
Continuous
Alternately increasing and decreasing in periodic waves
Symmetric with respect to the origin (odd)
Bounded
Absolute maximum of 1
Absolute minimum of -1
No horizontal asymptotes
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} \sin x$ and $\lim_{x \to \infty} \sin x$ do not exist. (The function values continually oscillate between -1 and 1 and approach no limit.)

We can add to this list that $y = \sin x$ is *periodic*, with period 2π. We can also add understanding of where the sine function comes from: By definition, $\sin t$ is the y-coordinate of the point P on the unit circle to which the real number t gets wrapped (or, equivalently, the point P on the unit circle determined by an angle of t radians in standard position). In fact, now we can see where the wavy graph comes from. Try Exploration 1.

EXPLORATION 1 Graphing $\sin t$ as a Function of t

Set your grapher to radian mode, parametric, and "simultaneous" graphing modes.

Set Tmin $= 0$, Tmax $= 6.3$, Tstep $= \pi/24$.

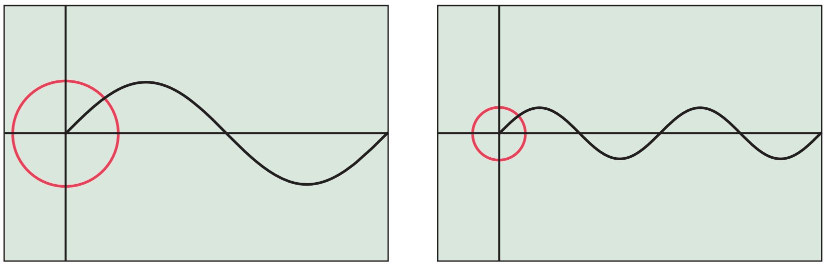
Set the (x, y) window to $[-1.2, 6.3]$ by $[-2.5, 2.5]$.

Set $X_{1T} = \cos(T)$ and $Y_{1T} = \sin(T)$. This will graph the unit circle. Set $X_{2T} = T$ and $Y_{2T} = \sin(T)$. This will graph $\sin(T)$ as a function of T.

Now start the graph and watch the point go counterclockwise around the unit circle as t goes from 0 to 2π in the positive direction. You will simultaneously see the y-coordinate of the point being graphed as a function of t along the horizontal t-axis. You can clear the drawing and watch the graph as many times as you need to in order to answer the following questions.

1. Where is the point on the unit circle when the wave is at its highest?
2. Where is the point on the unit circle when the wave is at its lowest?
3. Why do both graphs cross the x-axis at the same time?
4. Double the value of Tmax and change the window to $[-2.4, 12.6]$ by $[-5, 5]$. If your grapher can change "style" to show a moving point, choose that style for the unit circle graph. Run the graph and watch how the sine curve tracks the y-coordinate of the point as it moves around the unit circle.
5. Explain from what you have seen why the period of the sine function is 2π.
6. Challenge: Can you modify the grapher settings to show dynamically how the cosine function tracks the x-coordinate as the point moves around the unit circle?

Although a static picture does not do the dynamic simulation justice, Figure 4.38 shows the final screens for the two graphs in Exploration 1.

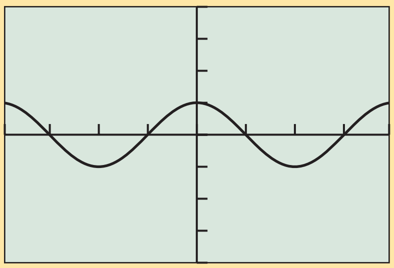


[–1.2, 6.3] by [–2.5, 2.5]
(a)

[−2.4, 12.6] by [−5, 5]
(b)

FIGURE 4.38 The graph of $y = \sin t$ tracks the y-coordinate of the point determined by t as it moves around the unit circle.

BASIC FUNCTION The Cosine Function



$[-2\pi, 2\pi]$ by [–4, 4]

FIGURE 4.38A

$f(x) = \cos x$
Domain: All reals
Range: $[-1, 1]$
Continuous
Alternately increasing and decreasing in periodic waves
Symmetric with respect to the y-axis (even)
Bounded
Absolute maximum of 1
Absolute minimum of -1
No horizontal asymptotes
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} \cos x$ and $\lim_{x \to \infty} \cos x$ do not exist. (The function values continually oscillate between -1 and 1 and approach no limit.)

As with the sine function, we can add the observation that it is periodic, with period 2π.

Sinusoids and Transformations

A comparison of the graphs of $y = \sin x$ and $y = \cos x$ suggests that either one can be obtained from the other by a horizontal translation (Section 1.5). In fact, we will prove later in this section that $\cos x = \sin(x + \pi/2)$. Each graph is an example of a *sinusoid*. In general, any transformation of a sine function (or the graph of such a function) is a sinusoid.

DEFINITION Sinusoid

A function is a **sinusoid** if it can be written in the form

$$f(x) = a \sin(bx + c) + d$$

where a, b, c, and d are constants and neither a nor b is 0.

Since cosine functions are themselves translations of sine functions, any transformation of a cosine function is also a sinusoid by the above definition.

There is a special vocabulary used to describe some of our usual graphical transformations when we apply them to sinusoids. Horizontal stretches and shrinks affect the *period* and the *frequency*, vertical stretches and shrinks affect the *amplitude*, and horizontal translations bring about *phase shifts*. All of these terms are associated with waves, and waves are quite naturally associated with sinusoids.

DEFINITION Amplitude of a Sinusoid

The **amplitude** of the sinusoid $f(x) = a \sin(bx + c) + d$ is $|a|$. Similarly, the amplitude of $f(x) = a \cos(bx + c) + d$ is $|a|$.

Graphically, the amplitude is half the height of the wave.

EXAMPLE 1 Vertical Stretch or Shrink and Amplitude

Find the amplitude of each function and use the language of transformations to describe how the graphs are related.

(a) $y_1 = \cos x$ **(b)** $y_2 = \frac{1}{2} \cos x$ **(c)** $y_3 = -3 \cos x$

SOLUTION

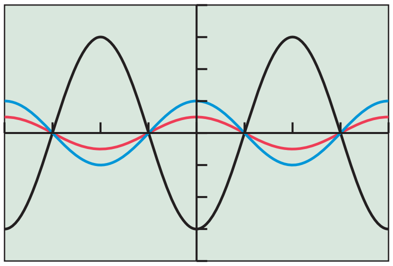
Solve Algebraically The amplitudes are (a) 1, (b) 1/2, and (c) $|-3| = 3$.

The graph of y_2 is a vertical shrink of the graph of y_1 by a factor of 1/2.

The graph of y_3 is a vertical stretch of the graph of y_1 by a factor of 3, and a reflection across the x-axis, performed in either order. (We do not call this a vertical stretch by a factor of -3, nor do we say that the amplitude is -3.)

Support Graphically The graphs of the three functions are shown in Figure 4.39. You should be able to tell which is which quite easily by checking the amplitudes.

Now try Exercise 1.



$[-2\pi, 2\pi]$ by $[-4, 4]$

FIGURE 4.39 Sinusoids (in this case, cosine curves) of different amplitudes. (Example 1)

You learned in Section 1.5 that the graph of $y = f(bx)$ when $|b| > 1$ is a horizontal shrink of the graph of $y = f(x)$ by a factor of $1/|b|$. That is exactly what happens with sinusoids, but we can add the observation that the period shrinks by the same factor. When $|b| < 1$, the effect on both the graph and the period is a horizontal stretch by a factor of $1/|b|$, plus a reflection across the y-axis if $b < 0$.

Period of a Sinusoid

The period of the sinusoid $f(x) = a \sin(bx + c) + d$ is $2\pi/|b|$. Similarly, the period of $f(x) = a \cos(bx + c) + d$ is $2\pi/|b|$.

Graphically, the period is the length of one full cycle of the wave.

EXAMPLE 2 Horizontal Stretch or Shrink and Period

Find the period of each function and use the language of transformations to describe how the graphs are related.

(a) $y_1 = \sin x$ **(b)** $y_2 = -2 \sin\left(\frac{x}{3}\right)$ **(c)** $y_3 = 3 \sin(-2x)$

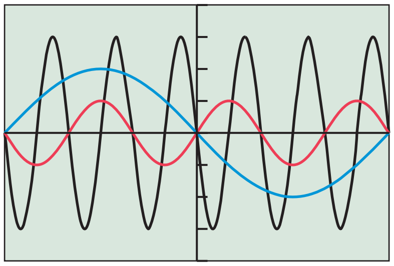
SOLUTION

Solve Algebraically The periods are (a) 2π, (b) $2\pi/(1/3) = 6\pi$, and (c) $2\pi/|-2| = \pi$.

The graph of y_2 is a horizontal stretch of the graph of y_1 by a factor of 3, a vertical stretch by a factor of 2, and a reflection across the x-axis, performed in any order.

The graph of y_3 is a horizontal shrink of the graph of y_1 by a factor of 1/2, a vertical stretch by a factor of 3, and a reflection across the y-axis, performed in any order. (Note that we do not call this a horizontal shrink by a factor of $-1/2$, nor do we say that the period is $-\pi$.)

Support Graphically The graphs of the three functions are shown in Figure 4.40. You should be able to tell which is which quite easily by checking the periods or the amplitudes. ***Now try Exercise 9.***



$[-3\pi, 3\pi]$ by $[-4, 4]$

FIGURE 4.40 Sinusoids (in this case, sine curves) of different amplitudes and periods. (Example 2)

In some applications, the *frequency* of a sinusoid is an important consideration. The frequency is simply the reciprocal of the period.

Frequency of a Sinusoid

The **frequency** of the sinusoid $f(x) = a \sin(bx + c) + d$ is $|b|/2\pi$. Similarly, the frequency of $f(x) = a \cos(bx + c) + d$ is $|b|/2\pi$.

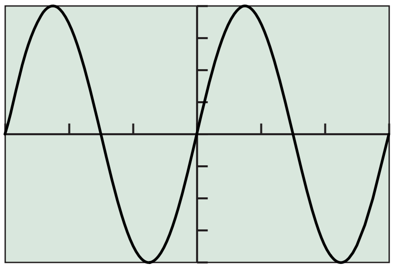
Graphically, the frequency is the number of complete cycles the wave completes in a unit interval.

EXAMPLE 3 Finding the Frequency of a Sinusoid

Find the frequency of the function $f(x) = 4 \sin(2x/3)$ and interpret its meaning graphically.

Sketch the graph in the window $[-3\pi, 3\pi]$ by $[-4, 4]$.

SOLUTION The frequency is $(2/3) \div 2\pi = 1/(3\pi)$. This is the reciprocal of the period, which is 3π. The graphical interpretation is that the graph completes 1 full cycle per interval of length 3π. (That, of course, is what having a period of 3π is all about.) The graph is shown in Figure 4.41. ***Now try Exercise 17.***



$[-3\pi, 3\pi]$ by $[-4, 4]$

FIGURE 4.41 The graph of the function $f(x) = 4 \sin(2x/3)$. It has frequency $1/(3\pi)$, so it completes 1 full cycle per interval of length 3π. (Example 3)

Recall from Section 1.5 that the graph of $y = f(x + c)$ is a translation of the graph of $y = f(x)$ by c units to the left when $c > 0$. That is exactly what happens with sinusoids, but using terminology with its roots in electrical engineering, we say that the wave undergoes a **phase shift** of $-c$.

EXAMPLE 4 Getting One Sinusoid from Another by a Phase Shift

(a) Write the cosine function as a phase shift of the sine function.

(b) Write the sine function as a phase shift of the cosine function.

SOLUTION

(a) The function $y = \sin x$ has a maximum at $x = \pi/2$, while the function $y = \cos x$ has a maximum at $x = 0$. Therefore, we need to shift the sine curve $\pi/2$ units to the *left* to get the cosine curve:

$$\cos x = \sin (x + \pi/2)$$

(b) It follows from the work in (a) that we need to shift the cosine curve $\pi/2$ units to the right to get the sine curve:

$$\sin x = \cos (x - \pi/2)$$

You can support with your grapher that these statements are true. Incidentally, there are many other translations that would have worked just as well. Adding any integral multiple of 2π to the phase shift would result in the same graph.

Now try Exercise 41.

One note of caution applies when combining these transformations. A horizontal stretch or shrink affects the variable along the horizontal axis, so it *also affects the phase shift*. Consider the transformation in Example 5.

EXAMPLE 5 Combining a Phase Shift with a Period Change

Construct a sinusoid with period $\pi/5$ and amplitude 6 that goes through $(2, 0)$.

SOLUTION To find the coefficient of x, we set $2\pi/|b| = \pi/5$ and solve to find that $b = \pm 10$. We arbitrarily choose $b = 10$. (Either will satisfy the specified conditions.)

For amplitude 6, we have $|a| = 6$. Again, we arbitrarily choose the positive value. The graph of $y = 6 \sin (10x)$ has the required amplitude and period, but it does not go through the point $(2, 0)$. It does, however, go through the point $(0, 0)$, so all that is needed is a phase shift of $+2$ to finish our function. Replacing x by $x - 2$, we get

$$y = 6 \sin (10(x - 2)) = 6 \sin (10x - 20).$$

Notice that we did *not* get the function $y = 6 \sin (10x - 2)$. That function would represent a phase shift of $y = \sin (10x)$, but only by 2/10, not 2. Parentheses are important when combining phase shifts with horizontal stretches and shrinks.

Now try Exercise 59.

Graphs of Sinusoids

The graphs of $y = a \sin (b(x - h)) + k$ and $y = a \cos (b(x - h)) + k$ (where $a \neq 0$ and $b \neq 0$) have the following characteristics:

$$\text{amplitude} = |a|;$$

$$\text{period} = \frac{2\pi}{|b|};$$

$$\text{frequency} = \frac{|b|}{2\pi}.$$

When compared to the graphs of $y = a \sin bx$ and $y = a \cos bx$, respectively, they also have the following characteristics:

a phase shift of h; a vertical translation of k.

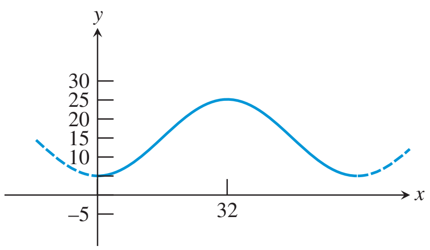
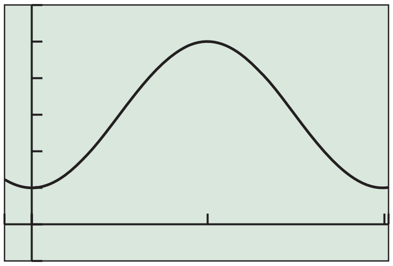


FIGURE 4.42 A sinusoid with specifications. (Example 6)



[–5, 65] by [–5, 30]

FIGURE 4.43 The graph of the function $y = -10 \cos((\pi/32)x) + 15$. (Example 8)

EXAMPLE 6 Constructing a Sinusoid by Transformations

Construct a sinusoid $y = f(x)$ that rises from a minimum value of $y = 5$ at $x = 0$ to a maximum value of $y = 25$ at $x = 32$. (See Figure 4.42.)

SOLUTION

Solve Algebraically The amplitude of this sinusoid is half the height of the graph: $(25 - 5)/2 = 10$. So $|a| = 10$. The period is 64 (since a full period goes from minimum to maximum and back down to the minimum). So set $2\pi/|b| = 64$. Solving, we get $|b| = \pi/32$.

We need a sinusoid that takes on its minimum value at $x = 0$. We could shift the graph of sine or cosine horizontally, but it is easier to take the cosine curve (which assumes its *maximum* value at $x = 0$) and turn it upside down. This reflection can be obtained by letting $a = -10$ rather than 10.

So far we have:

$$y = -10 \cos\left(\pm\frac{\pi}{32}x\right)$$

$$= -10 \cos\left(\frac{\pi}{32}x\right) \qquad \text{(Since cos is an even function)}$$

Finally, we note that this function ranges from a minimum of -10 to a maximum of 10. We shift the graph vertically by 15 to obtain a function that ranges from a minimum of 5 to a maximum of 25, as required. Thus

$$y = -10 \cos\left(\frac{\pi}{32}x\right) + 15.$$

Support Graphically We support our answer graphically by graphing the function (Figure 4.43). ***Now try Exercise 69.***

Modeling Periodic Behavior with Sinusoids

Example 6 was intended as more than just a review of the graphical transformations. Constructing a sinusoid with specific properties is often the key step in modeling physical situations that exhibit periodic behavior over time. The procedure we followed in Example 6 can be summarized as follows:

Constructing a Sinusoidal Model Using Time

1. Determine the maximum value M and minimum value m. The amplitude A of the sinusoid will be $A = \frac{M - m}{2}$, and the vertical shift will be $C = \frac{M + m}{2}$.
2. Determine the period p, the time interval of a single cycle of the periodic function. The horizontal shrink (or stretch) will be $B = \frac{2\pi}{p}$.
3. Choose an appropriate sinusoid based on behavior at some given time T. For example, at time T:

   $f(t) = A \cos(B(t - T)) + C$ attains a maximum value;

   $f(t) = -A \cos(B(t - T)) + C$ attains a minimum value;

   $f(t) = A \sin(B(t - T)) + C$ is halfway between a minimum and a maximum value;

   $f(t) = -A \sin(B(t - T)) + C$ is halfway between a maximum and a minimum value.

We apply the procedure in Example 7 to model the ebb and flow of a tide.

EXAMPLE 7 Calculating the Ebb and Flow of Tides

One particular July 4th in Galveston, TX, high tide occurred at 9:36 A.M. At that time the water at the end of the 61st Street Pier was 2.7 meters deep. Low tide occurred at 3:48 P.M., at which time the water was only 2.1 meters deep. Assume that the depth of the water is a sinusoidal function of time with a period of half a lunar day (about 12 hours 24 minutes).

(a) At what time on the 4th of July did the first low tide occur?

(b) What was the approximate depth of the water at 6:00 A.M. and at 3:00 P.M. that day?

(c) What was the first time on July 4th when the water was 2.4 meters deep?

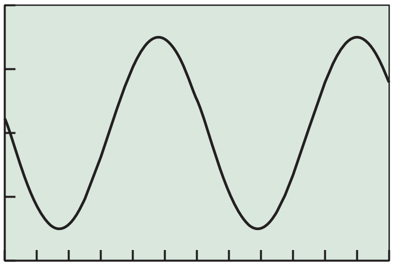
SOLUTION

Model We want to model the depth D as a sinusoidal function of time t. The depth varies from a maximum of 2.7 meters to a minimum of 2.1 meters, so the amplitude $A = \frac{2.7 - 2.1}{2} = 0.3$, and the vertical shift will be $C = \frac{2.7 + 2.1}{2} = 2.4$. The period is 12 hours 24 minutes, which converts to 12.4 hours, so $B = \frac{2\pi}{12.4} = \frac{\pi}{6.2}$.

We need a sinusoid that assumes its maximum value at 9:36 A.M. (which converts to 9.6 hours after midnight, a convenient time 0). We choose the cosine model. Thus,

$$D(t) = 0.3 \cos\left(\frac{\pi}{6.2}(t - 9.6)\right) + 2.4.$$

Solve Graphically The graph over the 24-hour period of July 4th is shown in Figure 4.44.



[0, 24] by [2, 2.8]

FIGURE 4.44 The Galveston tide graph. (Example 7)

We now use the graph to answer the questions posed.

(a) The first low tide corresponds to the first local minimum on the graph. We find graphically that this occurs at $t = 3.4$. This translates to $3 + (0.4)(60) = 3{:}24$ A.M.

(b) The depth at 6:00 A.M. is $D(6) \approx 2.32$ meters. The depth at 3:00 P.M. is $D(12 + 3) = D(15) \approx 2.12$ meters.

(c) The first time the water is 2.4 meters deep corresponds to the leftmost intersection of the sinusoid with the line $y = 2.4$. We use the grapher to find that $t = 0.3$. This translates to $0 + (0.3)(60) = 00{:}18$ A.M., which we write as 12:18 A.M.

Now try Exercise 75.

We will see more applications of this kind when we look at *simple harmonic motion* in Section 4.8.

QUICK REVIEW 4.4 *(For help, go to Sections 1.6, 4.1, and 4.2.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–3, state the sign (positive or negative) of the function in each quadrant.

1. $\sin x$ **2.** $\cos x$

3. $\tan x$

In Exercises 4–6, give the radian measure of the angle.

4. 135° **5.** −150° **6.** 450°

In Exercises 7–10, find a transformation that will transform the graph of y_1 to the graph of y_2.

7. $y_1 = \sqrt{x}$ and $y_2 = 3\sqrt{x}$

8. $y_1 = e^x$ and $y_2 = e^{-x}$

9. $y_1 = \ln x$ and $y_2 = 0.5 \ln x$

10. $y_1 = x^3$ and $y_2 = x^3 - 2$

SECTION 4.4 EXERCISES

In Exercises 1–6, find the amplitude of the function and use the language of transformations to describe how the graph of the function is related to the graph of $y = \sin x$.

1. $y = 2 \sin x$

2. $y = \frac{2}{3} \sin x$

3. $y = -4 \sin x$

4. $y = -\frac{7}{4} \sin x$

5. $y = 0.73 \sin x$

6. $y = -2.34 \sin x$

In Exercises 7–12, find the period of the function and use the language of transformations to describe how the graph of the function is related to the graph of $y = \cos x$.

7. $y = \cos 3x$

8. $y = \cos x/5$

9. $y = \cos(-7x)$

10. $y = \cos(-0.4x)$

11. $y = 3 \cos 2x$

12. $y = \frac{1}{4} \cos \frac{2x}{3}$

In Exercises 13–16, find the amplitude, period, and frequency of the function and use this information (not your calculator) to sketch a graph of the function in the window $[-3\pi, 3\pi]$ by $[-4, 4]$.

13. $y = 3 \sin \frac{x}{2}$

14. $y = 2 \cos \frac{x}{3}$

15. $y = -\frac{3}{2} \sin 2x$

16. $y = -4 \sin \frac{2x}{3}$

In Exercises 17–22, graph one period of the function. Use your understanding of transformations, not your graphing calculators. Be sure to show the scale on both axes.

17. $y = 2 \sin x$

18. $y = 2.5 \sin x$

19. $y = 3 \cos x$

20. $y = -2 \cos x$

21. $y = -0.5 \sin x$

22. $y = 4 \cos x$

In Exercises 23–28, graph three periods of the function. Use your understanding of transformations, not your graphing calculators. Be sure to show the scale on both axes.

23. $y = 5 \sin 2x$

24. $y = 3 \cos \frac{x}{2}$

25. $y = 0.5 \cos 3x$

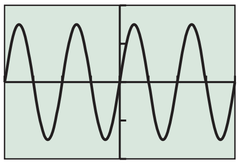
26. $y = 20 \sin 4x$

27. $y = 4 \sin \frac{x}{4}$

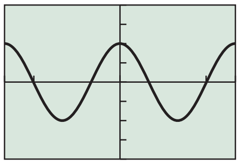
28. $y = 8 \cos 5x$

In Exercises 29–34, specify the period and amplitude of each function. Then give the viewing window in which the graph is shown. Use your understanding of transformations, not your graphing calculators.

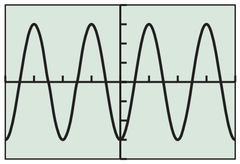
29. $y = 1.5 \sin 2x$



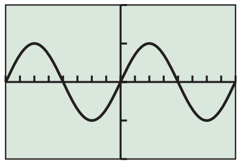
30. $y = 2 \cos 3x$



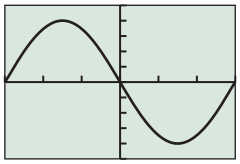
31. $y = -3 \cos 2x$



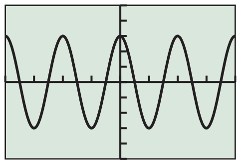
32. $y = 5 \sin \frac{x}{2}$



33. $y = -4 \sin \frac{\pi}{3} x$



34. $y = 3 \cos \pi x$



In Exercises 35–40, identify the maximum and minimum values and the zeros of the function in the interval $[-2\pi, 2\pi]$. Use your understanding of transformations, not your graphing calculators.

35. $y = 2 \sin x$

36. $y = 3 \cos \frac{x}{2}$

37. $y = \cos 2x$

38. $y = \frac{1}{2} \sin x$

39. $y = -\cos 2x$

40. $y = -2 \sin x$

41. Write the function $y = -\sin x$ as a phase shift of $y = \sin x$.

42. Write the function $y = -\cos x$ as a phase shift of $y = \sin x$.

In Exercises 43–48, describe the transformations required to obtain the graph of the given function from a basic trigonometric graph.

43. $y = 0.5 \sin 3x$

44. $y = 1.5 \cos 4x$

45. $y = -\frac{2}{3} \cos \frac{x}{3}$

46. $y = \frac{3}{4} \sin \frac{x}{5}$

47. $y = 3 \cos \frac{2\pi x}{3}$

48. $y = -2 \sin \frac{\pi x}{4}$

In Exercises 49–52, describe the transformations required to obtain the graph of y_2 from the graph of y_1.

49. $y_1 = \cos 2x$ and $y_2 = \frac{5}{3} \cos 2x$

50. $y_1 = 2 \cos\left(x + \frac{\pi}{3}\right)$ and $y_2 = \cos\left(x + \frac{\pi}{4}\right)$

51. $y_1 = 2 \cos \pi x$ and $y_2 = 2 \cos 2\pi x$

52. $y_1 = 3 \sin \frac{2\pi x}{3}$ and $y_2 = 2 \sin \frac{\pi x}{3}$

In Exercises 53–56, select the pair of functions that have identical graphs.

53. **(a)** $y = \cos x$ **(b)** $y = \sin\left(x + \frac{\pi}{2}\right)$

(c) $y = \cos\left(x + \frac{\pi}{2}\right)$

54. **(a)** $y = \sin x$ **(b)** $y = \cos\left(x - \frac{\pi}{2}\right)$

(c) $y = \cos x$

55. **(a)** $y = \sin\left(x + \frac{\pi}{2}\right)$ **(b)** $y = -\cos(x - \pi)$

(c) $y = \cos\left(x - \frac{\pi}{2}\right)$

56. **(a)** $y = \sin\left(2x + \frac{\pi}{4}\right)$ **(b)** $y = \cos\left(2x - \frac{\pi}{2}\right)$

(c) $y = \cos\left(2x - \frac{\pi}{4}\right)$

In Exercises 57–60, construct a sinusoid with the given amplitude and period that goes through the given point.

57. Amplitude 3, period π, point $(0, 0)$

58. Amplitude 2, period 3π, point $(0, 0)$

59. Amplitude 1.5, period $\pi/6$, point $(1, 0)$

60. Amplitude 3.2, period $\pi/7$, point $(5, 0)$

In Exercises 61–68, state the amplitude and period of the sinusoid, and (relative to the basic function) the phase shift and vertical translation.

61. $y = -2\sin\left(x - \frac{\pi}{4}\right) + 1$

62. $y = -3.5\sin\left(2x - \frac{\pi}{2}\right) - 1$

63. $y = 5\cos\left(3x - \frac{\pi}{6}\right) + 0.5$

64. $y = 3\cos(x + 3) - 2$

65. $y = 2\cos 2\pi x + 1$

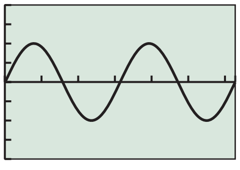
66. $y = 4\cos 3\pi x - 2$

67. $y = \frac{7}{3}\sin\left(x + \frac{5}{2}\right) - 1$

68. $y = \frac{2}{3}\cos\left(\frac{x - 3}{4}\right) + 1$

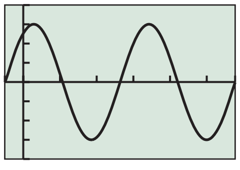
In Exercises 69 and 70, find values a, b, h, and k so that the graph of the function $y = a\sin(b(x + h)) + k$ is the curve shown.

69.



[0, 6.28] by [–4, 4]

70.



[–0.5, 5.78] by [–4, 4]

71. **Points of Intersection** Graph $y = 1.3^{-x}$ and $y = 1.3^{-x}\cos x$ for x in the interval $[-1, 8]$.

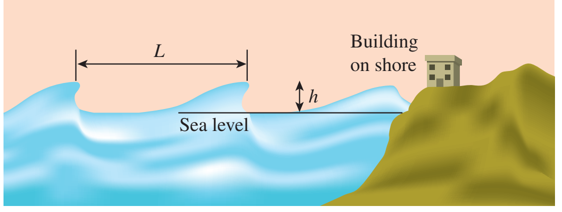
(a) How many points of intersection do there appear to be?

(b) Find the coordinates of each point of intersection.

72. **Motion of a Buoy** A signal buoy in the Chesapeake Bay bobs up and down with the height h of its transmitter (in feet) above sea level modeled by $h = a\sin bt + 5$. During a small squall its height varies from 1 ft to 9 ft and there are 3.5 sec from one 9-ft height to the next. What are the values of the constants a and b?

73. **Ferris Wheel** A Ferris wheel 50 ft in diameter makes one revolution every 40 sec. If the center of the wheel is 30 ft above the ground, how long after reaching the low point is a rider 50 ft above the ground?

74. **Tsunami Wave** An earthquake occurred at 9:40 A.M. on Nov. 1, 1755, at Lisbon, Portugal, and started a *tsunami* (often called a tidal wave) in the ocean. It produced waves that traveled more than 540 ft/sec (370 mph) and reached a height of 60 ft. If the period of the waves was 30 min or 1800 sec, estimate the length L between the crests.



75. **Ebb and Flow** On a particular Labor Day, the high tide in Southern California occurs at 7:12 A.M. At that time you measure the water at the end of the Santa Monica Pier to be 11 ft deep. At 1:24 P.M. it is low tide, and you measure the water to be only 7 ft deep. Assume the depth of the water is a sinusoidal function of time with a period of 1/2 a lunar day, which is about 12 hr 24 min.

(a) At what time on that Labor Day does the first low tide occur?

(b) What was the approximate depth of the water at 4:00 A.M. and at 9:00 P.M.?

(c) What is the first time on that Labor Day that the water is 9 ft deep?

76. **Blood Pressure** The function

$$P = 120 + 30\sin 2\pi t$$

models the blood pressure (in millimeters of mercury) for a person who has a (high) blood pressure of 150/90; t represents seconds.

(a) What is the period of this function?

(b) How many heartbeats are there each minute?

(c) Graph this function to model a 10-sec time interval.

77. **Bouncing Block** A block mounted on a spring is set into motion directly above a motion detector, which registers the distance to the block at intervals of 0.1 second. When the block is released, it is 7.2 cm above the motion detector. The table below shows the data collected by the motion detector during the first two seconds, with distance d measured in centimeters:

(a) Make a scatter plot of d as a function of t and estimate the maximum d visually. Use this number and the given minimum (7.2) to compute the amplitude of the block's motion.

(b) Estimate the period of the block's motion visually from the scatter plot.

(c) Model the motion of the block as a sinusoidal function $d(t)$.

(d) Graph your function with the scatter plot to support your model graphically.

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
d	9.2	13.9	18.8	21.4	20.0	15.6	10.5	7.4	8.1	12.1

t	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
d	17.3	20.8	20.8	17.2	12.0	8.1	7.5	10.5	15.6	19.9

78. LP Turntable A suction-cup-tipped arrow is secured vertically to the outer edge of a turntable designed for playing LP phonograph records (ask your parents). A motion detector is situated 60 cm away. The turntable is switched on and a motion detector measures the distance to the arrow as it revolves around the turntable. The table below shows the distance d as a function of time during the first 4 seconds.

t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
d	63.5	71.6	79.8	84.7	84.7	79.8	71.6	63.5	60.0	63.5

t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
d	71.6	79.8	84.7	84.7	79.8	71.6	63.5	60.0	63.5	71.6

(a) If the turntable is 25.4 cm in diameter, find the amplitude of the arrow's motion.

(b) Find the period of the arrow's motion by analyzing the data.

(c) Model the motion of the arrow as a sinusoidal function $d(t)$.

(d) Graph your function with a scatter plot to support your model graphically.

79. Temperature Data The normal monthly Fahrenheit temperatures in Albuquerque, NM, are shown in the table below (month 1 = Jan, month 2 = Feb, etc.):

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	36	41	48	56	65	75	79	76	69	57	44	36

Source: National Climatic Data Center, as reported in The World Almanac and Book of Facts 2009.

Model the temperature T as a sinusoidal function of time, using 36 as the minimum value and 79 as the maximum value. Support your answer graphically by graphing your function with a scatter plot.

80. Temperature Data The normal monthly Fahrenheit temperatures in Helena, MT, are shown in the table below (month 1 = Jan, month 2 = Feb, etc.):

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	20	26	35	44	53	61	68	67	56	45	31	21

Source: National Climatic Data Center, as reported in The World Almanac and Book of Facts 2009.

Model the temperature T as a sinusoidal function of time, using 20 as the minimum value and 68 as the maximum value. Support your answer graphically by graphing your function with a scatter plot.

Standardized Test Questions

81. True or False The graph of $y = \sin 2x$ has half the period of the graph of $y = \sin 4x$. Justify your answer.

82. True or False Every sinusoid can be written as $y = A \cos (Bx + C)$ for some real numbers A, B, and C. Justify your answer.

You may use a graphing calculator when answering these questions.

83. Multiple Choice A sinusoid with amplitude 4 has a minimum value of 5. Its maximum value is

(A) 7. **(B)** 9. **(C)** 11.

(D) 13. **(E)** 15.

84. Multiple Choice The graph of $y = f(x)$ is a sinusoid with period 45 passing through the point (6, 0). Which of the following can be determined from the given information?

I. $f(0)$ **II.** $f(6)$ **III.** $f(96)$

(A) I only **(B)** II only

(C) I and III only **(D)** II and III only

(E) I, II, and III only

85. Multiple Choice The period of the function $f(x) = 210 \sin (420x + 840)$ is

(A) $\pi/840$. **(B)** $\pi/420$. **(C)** $\pi/210$.

(D) $210/\pi$. **(E)** $420/\pi$.

86. Multiple Choice The number of solutions to the equation $\sin (2000x) = 3/7$ in the interval $[0, 2\pi]$ is

(A) 1000. **(B)** 2000. **(C)** 4000.

(D) 6000. **(E)** 8000.

Explorations

87. Approximating Cosine

(a) Draw a scatter plot $(x, \cos x)$ for the 17 special angles x, where $-\pi \le x \le \pi$.

(b) Find a quartic regression for the data.

(c) Compare the approximation to the cosine function given by the quartic regression with the Taylor polynomial approximations given in Exercise 80 of Section 4.3.

88. Approximating Sine

(a) Draw a scatter plot $(x, \sin x)$ for the 17 special angles x, where $-\pi \le x \le \pi$.

(b) Find a cubic regression for the data.

(c) Compare the approximation to the sine function given by the cubic regression with the Taylor polynomial approximations given in Exercise 79 of Section 4.3.

89. Visualizing a Musical Note A piano tuner strikes a tuning fork for the note middle C and creates a sound wave that can be modeled by

$$y = 1.5 \sin 524\pi t,$$

where t is the time in seconds.

(a) What is the period p of this function?

(b) What is the frequency $f = 1/p$ of this note?

(c) Graph the function.

90. Writing to Learn In a certain video game a cursor bounces back and forth horizontally across the screen at a constant rate. Its distance d from the center of the screen varies with time t and hence can be described as a function of t. Explain why this horizontal distance d from the center of the screen *does not vary* according to an equation $d = a \sin bt$, where t represents seconds. You may find it helpful to include a graph in your explanation.

91. Group Activity Using only integer values of a and b between 1 and 9 inclusive, look at graphs of functions of the form

$$y = \sin(ax)\cos(bx) - \cos(ax)\sin(bx)$$

for various values of a and b. (A group can look at more graphs at a time than one person can.)

(a) Some values of a and b result in the graph of $y = \sin x$. Find a general rule for such values of a and b.

(b) Some values of a and b result in the graph of $y = \sin 2x$. Find a general rule for such values of a and b.

(c) Can you guess which values of a and b will result in the graph of $y = \sin kx$ for an arbitrary integer k?

92. Group Activity Using only integer values of a and b between 1 and 9 inclusive, look at graphs of functions of the form

$$y = \cos(ax)\cos(bx) + \sin(ax)\sin(bx)$$

for various values of a and b. (A group can look at more graphs at a time than one person can.)

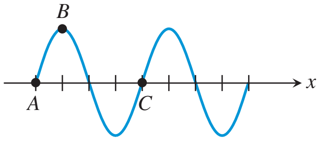
(a) Some values of a and b result in the graph of $y = \cos x$. Find a general rule for such values of a and b.

(b) Some values of a and b result in the graph of $y = \cos 2x$. Find a general rule for such values of a and b.

(c) Can you guess which values of a and b will result in the graph of $y = \cos kx$ for an arbitrary integer k?

Extending the Ideas

In Exercises 93–96, the graphs of the sine and cosine functions are waveforms like the figure below. By correctly labeling the coordinates of points A, B, and C, you will get the graph of the function given.



93. $y = 3\cos 2x$ and $A = \left(-\frac{\pi}{4}, 0\right)$. Find B and C.

94. $y = 4.5 \sin\left(x - \frac{\pi}{4}\right)$ and $A = \left(\frac{\pi}{4}, 0\right)$. Find B and C.

95. $y = 2\sin\left(3x - \frac{\pi}{4}\right)$ and $A = \left(\frac{\pi}{12}, 0\right)$. Find B and C.

96. $y = 3\sin(2x - \pi)$, and A is the first x-intercept on the right of the y-axis. Find A, B, and C.

97. The Ultimate Sinusoidal Equation It is an interesting fact that any sinusoid can be written in the form

$$y = a \sin[b(x - H)] + k,$$

where both a and b are positive numbers.

(a) Explain why you can assume b is positive. [*Hint:* Replace b by $-B$ and simplify.]

(b) Use one of the horizontal translation identities to prove that the equation

$$y = a\cos[b(x - h)] + k$$

has the same graph as

$$y = a\sin[b(x - H)] + k$$

for a correctly chosen value of H. Explain how to choose H.

(c) Give a unit circle argument for the identity $\sin(\theta + \pi) = -\sin\theta$. Support your unit circle argument graphically.

(d) Use the identity from (c) to prove that

$$y = -a\sin[b(x - h)] + k,\ a > 0,$$

has the same graph as

$$y = a\sin[b(x - H)] + k,\ a > 0$$

for a correctly chosen value of H. Explain how to choose H.

(e) Combine your results from (a)–(d) to prove that any sinusoid can be represented by the equation

$$y = a\sin[b(x - H)] + k$$

where a and b are both positive.