4.3 Trigonometry Extended: The Circular Functions

What you'll learn about

- Trigonometric Functions of Any Angle
- Trigonometric Functions of Real Numbers
- Periodic Functions
- The 16-Point Unit Circle

… and why

Extending trigonometric functions beyond triangle ratios opens up a new world of applications.

Trigonometric Functions of Any Angle

We now extend the definitions of the six basic trigonometric functions beyond triangles so that we do not have to restrict our attention to acute angles, or even to positive angles.

In geometry we think of an angle as a union of two rays with a common vertex. Trigonometry takes a more dynamic view by thinking of an angle in terms of a rotating ray. The beginning position of the ray, the **initial side**, is rotated about its endpoint, called the **vertex**. The final position is called the **terminal side**. The **measure of an angle** is a number that describes the amount of rotation from the initial side to the terminal side of the angle. **Positive angles** are generated by counterclockwise rotations and **negative angles** are generated by clockwise rotations. Figure 4.19 shows an angle of measure α, where α is a positive number.

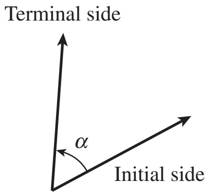


FIGURE 4.19 An angle with positive measure α.

To bring the power of coordinate geometry into the picture (literally), we usually place an angle in **standard position** in the Cartesian plane, with the vertex of the angle at the origin and its initial side lying along the positive x-axis. Figure 4.20 shows two angles in standard position, one with positive measure α and the other with negative measure β.

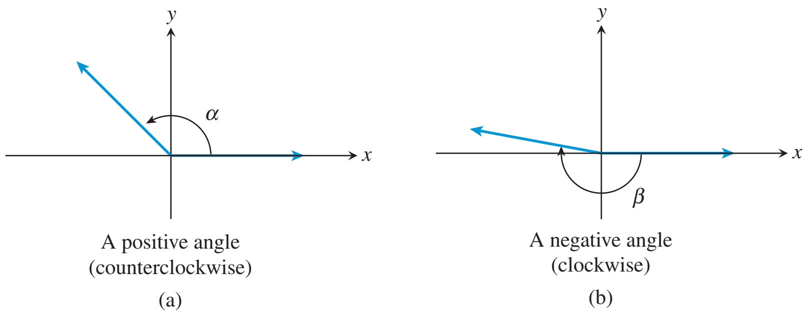
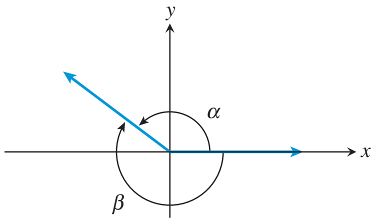


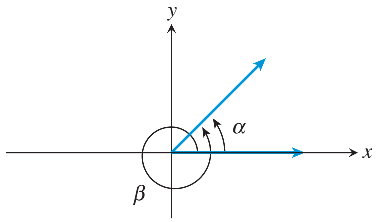
FIGURE 4.20 Two angles in standard position. In (a) the counterclockwise rotation generates an angle with positive measure α. In (b) the clockwise rotation generates an angle with negative measure β.

Two angles in this expanded angle-measurement system can have the same initial side and the same terminal side, yet have different measures. We call such angles **coterminal angles**. (See Figure 4.21 on the next page.) For example, angles of 90°, 450°, and −270° are all coterminal, as are angles of π radians, 3π radians, and -99π radians. In fact, angles are coterminal whenever they differ by an integer multiple of 360 degrees or by an integer multiple of 2π radians.



Positive and negative
coterminal angles

(a)



Two positive
coterminal angles

(b)

FIGURE 4.21 Coterminal angles. In (a) a positive angle and a negative angle are coterminal, while in (b) both coterminal angles are positive.

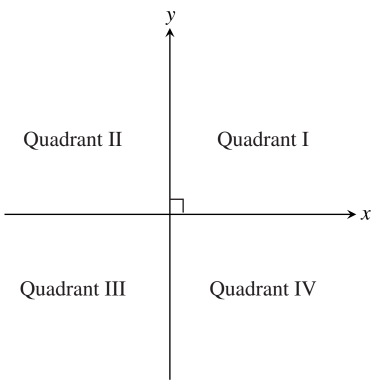


FIGURE 4.23 The four quadrants of the Cartesian plane. Both x and y are positive in QI (Quadrant I). Quadrants, like Super Bowls, are invariably designated by Roman numerals.

EXAMPLE 1 Finding Coterminal Angles

Find and draw a positive angle and a negative angle that are coterminal with the given angle.

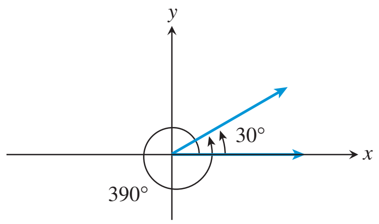
(a) $30°$ **(b)** $-150°$ **(c)** $\dfrac{2\pi}{3}$ radians

SOLUTION There are infinitely many possible solutions; we will show two for each angle.

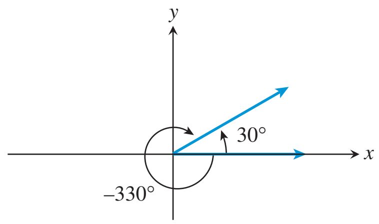
(a) Add $360°$: $30° + 360° = 390°$

Subtract $360°$: $30° - 360° = -330°$

Figure 4.22 shows these two angles, which are coterminal with the $30°$ angle.



(a)



(b)

FIGURE 4.22 Two angles coterminal with $30°$. (Example 1a)

(b) Add $360°$: $-150° + 360° = 210°$

Subtract $720°$: $-150° - 720° = -870°$

We leave it to you to draw the coterminal angles.

(c) Add 2π: $\dfrac{2\pi}{3} + 2\pi = \dfrac{2\pi}{3} + \dfrac{6\pi}{3} = \dfrac{8\pi}{3}$

Subtract 2π: $\dfrac{2\pi}{3} - 2\pi = \dfrac{2\pi}{3} - \dfrac{6\pi}{3} = -\dfrac{4\pi}{3}$

Again, we leave it to you to draw the coterminal angles.

Now try Exercise 1.

Extending the definitions of the six basic trigonometric functions so that they can apply to any angle is surprisingly easy, but first you need to see how our current definitions relate to the (x, y) coordinates in the Cartesian plane. We start in the first quadrant (see Figure 4.23), where the angles are all acute. Work through Exploration 1 before moving on.

EXPLORATION 1 Investigating First Quadrant Trigonometry

Let $P(x, y)$ be any point in the first quadrant (QI), and let r be the distance from P to the origin. (See Figure 4.24.)

1. Use the acute angle definition of the sine function (Section 4.2) to prove that $\sin\theta = y/r$.
2. Express $\cos\theta$ in terms of x and r.
3. Express $\tan\theta$ in terms of x and y.
4. Express the remaining three basic trigonometric functions in terms of x, y, and r.

If you have successfully completed Exploration 1, you should have no trouble verifying the solution to Example 2, which we show without the details.

EXAMPLE 2 Evaluating Trig Functions Determined by a Point in QI

Let θ be the acute angle in standard position whose terminal side contains the point $(5, 3)$. Find the six trigonometric functions of θ.

SOLUTION The distance from $(5, 3)$ to the origin is $\sqrt{34}$.

So
$$\sin\theta = \frac{3}{\sqrt{34}} \text{ or } \frac{3\sqrt{34}}{34} \qquad \csc\theta = \frac{\sqrt{34}}{3}$$
$$\cos\theta = \frac{5}{\sqrt{34}} \text{ or } \frac{5\sqrt{34}}{34} \qquad \sec\theta = \frac{\sqrt{34}}{5}$$
$$\tan\theta = \frac{3}{5} \qquad \cot\theta = \frac{5}{3}$$

Now try Exercise 5.

Now we have an easy way to extend the trigonometric functions to any angle: Use the same definitions in terms of x, y, and r—*whether or not x and y are positive*. Compare Example 3 to Example 2.

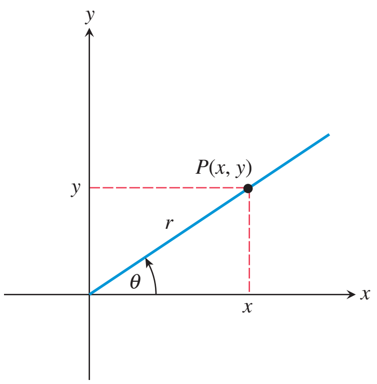


FIGURE 4.24 A point $P(x, y)$ in Quadrant I determines an acute angle θ. The number r denotes the distance from P to the origin. (Exploration 1)

EXAMPLE 3 Evaluating Trig Functions Determined by a Point Not in QI

Let θ be any angle in standard position whose terminal side contains the point $(-5, 3)$. Find the six trigonometric functions of θ.

SOLUTION The distance from $(-5, 3)$ to the origin is $\sqrt{34}$.

So
$$\sin\theta = \frac{3}{\sqrt{34}} \text{ or } \frac{3\sqrt{34}}{34} \qquad \csc\theta = \frac{\sqrt{34}}{3}$$
$$\cos\theta = \frac{-5}{\sqrt{34}} \text{ or } \frac{-5\sqrt{34}}{34} \qquad \sec\theta = \frac{\sqrt{34}}{-5}$$
$$\tan\theta = \frac{3}{-5} = -0.6 \qquad \cot\theta = \frac{-5}{3}$$

Now try Exercise 11.

Notice in Example 3 that θ is *any* angle in standard position whose terminal side contains the point $(-5, 3)$. There are infinitely many coterminal angles that could play the role of θ, some of them positive and some of them negative. The values of the six trigonometric functions would be the same for all of them.

We are now ready to state the formal definition.

DEFINITION Trigonometric Functions of any Angle

Let θ be any angle in standard position and let $P(x, y)$ be any point on the terminal side of the angle (except the origin). Let r denote the distance from $P(x, y)$ to the origin, i.e., let $r = \sqrt{x^2 + y^2}$. (See Figure 4.25.) Then

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}\ (y \neq 0)$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}\ (x \neq 0)$$
$$\tan\theta = \frac{y}{x}\ (x \neq 0) \qquad \cot\theta = \frac{x}{y}\ (y \neq 0)$$

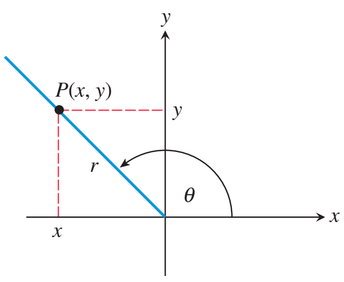


FIGURE 4.25 Defining the six trig functions of θ.

Examples 2 and 3 both began with a point $P(x, y)$ rather than an angle θ. Indeed, the point gave us so much information about the trigonometric ratios that we were able to compute them all without ever finding θ. So what do we do if we start with an angle θ in standard position and we want to evaluate the trigonometric functions? We try to find a point (x, y) on its terminal side. We illustrate this process with Example 4.

EXAMPLE 4 Evaluating the Trig Functions of 315°

Find the six trigonometric functions of 315°.

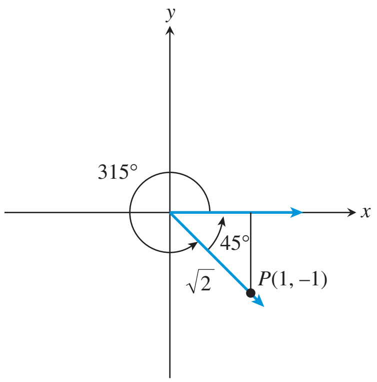


FIGURE 4.26 An angle of 315° in standard position determines a 45°–45°–90° *reference triangle.* (Example 4)

SOLUTION First we draw an angle of 315° in standard position. Without declaring a scale, pick a point P on the terminal side and connect it to the x-axis with a perpendicular segment. Notice that the triangle formed (called a **reference triangle**) is a 45°–45°–90° triangle. If we arbitrarily choose the horizontal and vertical sides of the reference triangle to be of length 1, then P has coordinates $(1, -1)$. (See Figure 4.26.)

We can now use the definitions with $x = 1$, $y = -1$, and $r = \sqrt{2}$.

$$\sin 315° = \frac{-1}{\sqrt{2}} \quad \text{or} \quad -\frac{\sqrt{2}}{2} \qquad \csc 315° = \frac{\sqrt{2}}{-1} = -\sqrt{2}$$

$$\cos 315° = \frac{1}{\sqrt{2}} \quad \text{or} \quad \frac{\sqrt{2}}{2} \qquad \sec 315° = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan 315° = \frac{-1}{1} = -1 \qquad \cot 315° = \frac{1}{-1} = -1$$

Now try Exercise 25.

The happy fact that the reference triangle in Example 4 was a 45°–45°–90° triangle enabled us to label a point P on the terminal side of the 315° angle and then to find the trigonometric function values. We would also be able to find P if the given angle were to produce a 30°–60°–90° reference triangle.

Evaluating Trig Functions of a Nonquadrantal Angle θ

1. Draw the angle θ in standard position, being careful to place the terminal side in the correct quadrant.
2. Without declaring a scale on either axis, label a point P (other than the origin) on the terminal side of θ.
3. Draw a perpendicular segment from P to the x-axis, determining the *reference triangle*. If this triangle is one of the triangles whose ratios you know, label the sides accordingly. If it is not, then you will need to use your calculator.
4. Use the sides of the triangle to determine the coordinates of point P, making them positive or negative according to the signs of x and y in that particular quadrant.
5. Use the coordinates of point P and the definitions to determine the six trig functions.

EXAMPLE 5 Evaluating More Trig Functions

Find the following without a calculator:

(a) $\sin(-210°)$

(b) $\tan(5\pi/3)$

(c) $\sec(-3\pi/4)$

(continued)

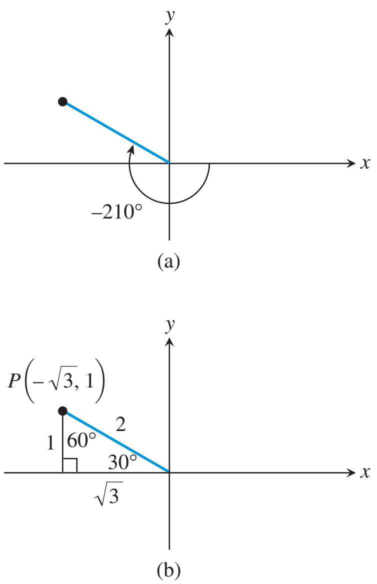


FIGURE 4.27 (Example 5a)

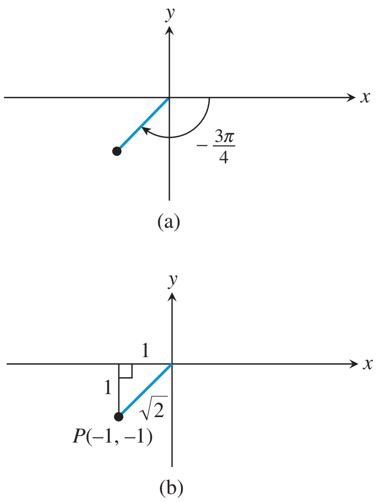


FIGURE 4.29 (Example 5c)

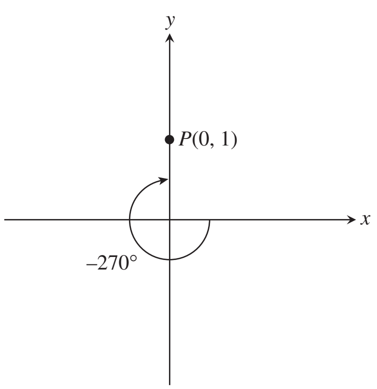


FIGURE 4.30 (Example 6a)

SOLUTION

(a) An angle of $-210°$ in standard position determines a 30°–60°–90° reference triangle in the second quadrant (Figure 4.27). We label the sides accordingly, then use the lengths of the sides to determine the point $P(-\sqrt{3}, 1)$. (Note that the x-coordinate is negative in the second quadrant.) The hypotenuse is $r = 2$. Therefore $\sin(-210°) = y/r = 1/2$.

(b) An angle of $5\pi/3$ radians in standard position determines a 30°–60°–90° reference triangle in the fourth quadrant (Figure 4.28). We label the sides accordingly, then use the lengths of the sides to determine the point $P(1, -\sqrt{3})$. (Note that the y-coordinate is negative in the fourth quadrant.) The hypotenuse is $r = 2$. Therefore $\tan(5\pi/3) = y/x = -\sqrt{3}/1 = -\sqrt{3}$.

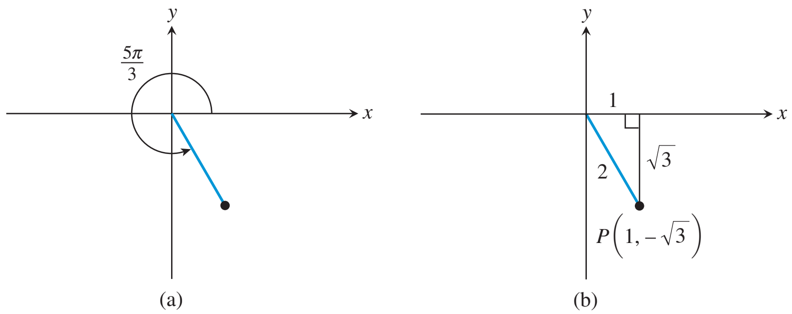


FIGURE 4.28 (Example 5b)

(c) An angle of $-3\pi/4$ radians in standard position determines a 45°–45°–90° reference triangle in the third quadrant. (See Figure 4.29.) We label the sides accordingly, then use the lengths of the sides to determine the point $P(-1, -1)$. (Note that both coordinates are negative in the third quadrant.) The hypotenuse is $r = \sqrt{2}$. Therefore $\sec(-3\pi/4) = r/x = \sqrt{2}/-1 = -\sqrt{2}$.

Now try Exercise 29.

Angles whose terminal sides lie along one of the coordinate axes are called **quadrantal angles**, and although they do not produce reference triangles at all, it is easy to pick a point P along one of the axes.

EXAMPLE 6 Evaluating Trig Functions of Quadrantal Angles

Find each of the following, if it exists. If the value is undefined, write "undefined."

(a) $\sin(-270°)$

(b) $\tan 3\pi$

(c) $\sec \dfrac{11\pi}{2}$

SOLUTION

(a) In standard position, the terminal side of an angle of $-270°$ lies along the positive y-axis (Figure 4.30). A convenient point P along the positive y-axis is the point for which $r = 1$, namely $(0, 1)$. Therefore

$$\sin(-270°) = \frac{y}{r} = \frac{1}{1} = 1.$$

Why Not Use a Calculator?

You might wonder why we would go through this procedure to produce values that could be found so easily with a calculator. The answer is to understand how trigonometry *works* in the coordinate plane. Ironically, technology has made these computational exercises more important than ever, since calculators have eliminated the need for the repetitive evaluations that once gave students their initial insights into the basic trig functions.

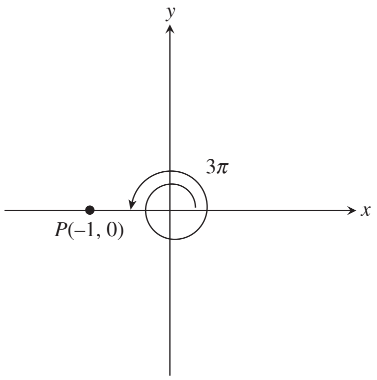


FIGURE 4.31 (Example 6b)

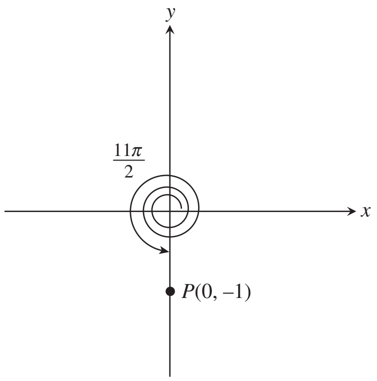


FIGURE 4.32 (Example 6c)

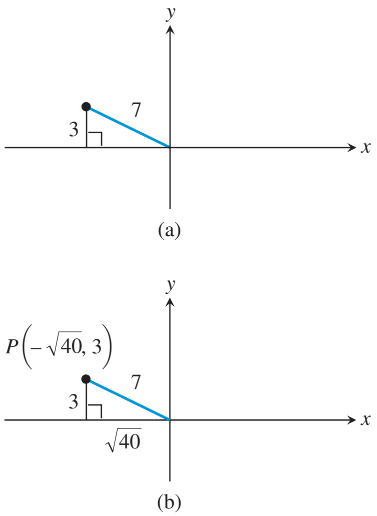


FIGURE 4.33 (Example 7a)

(b) In standard position, the terminal side of an angle of 3π lies along the negative x-axis. (See Figure 4.31.) A convenient point P along the negative x-axis is the point for which $r = 1$, namely $(-1, 0)$. Therefore

$$\tan 3\pi = \frac{y}{x} = \frac{0}{-1} = 0.$$

(c) In standard position, the terminal side of an angle of $11\pi/2$ lies along the negative y-axis. (See Figure 4.32.) A convenient point P along the negative y-axis is the point for which $r = 1$, namely $(0, -1)$. Therefore

$$\sec \frac{11\pi}{2} = \frac{r}{x} = \frac{1}{0}.$$

Now try Exercise 41.

Another good exercise is to use information from one trigonometric ratio to produce the other five. We do not need to know the angle θ, although we do need a hint as to the location of its terminal side so that we can sketch a reference triangle in the correct quadrant (or place a quadrantal angle on the correct side of the origin). Example 7 illustrates how this is done.

EXAMPLE 7 Using One Trig Ratio to Find the Others

Find $\cos\theta$ and $\tan\theta$ by using the given information to construct a reference triangle.

(a) $\sin\theta = \frac{3}{7}$ and $\tan\theta < 0$

(b) $\sec\theta = 3$ and $\sin\theta > 0$

(c) $\cot\theta$ is undefined and $\sec\theta$ is negative

SOLUTION

(a) Since $\sin\theta$ is positive, the terminal side is either in QI or in QII. The added fact that $\tan\theta$ is negative means that the terminal side is in QII. We draw a reference triangle in QII with $r = 7$ and $y = 3$ (Figure 4.33); then we use the Pythagorean Theorem to find that $x = -1\sqrt{7^2 - 3^2} = -\sqrt{40}$. (Note that x is negative in QII.)

We then use the definitions to get

$$\cos\theta = \frac{-\sqrt{40}}{7} \quad \text{and} \quad \tan\theta = \frac{3}{-\sqrt{40}} \quad \text{or} \quad \frac{-3\sqrt{10}}{20}.$$

(b) Since $\sec\theta$ is positive, the terminal side is either in QI or in QIV. The added fact that $\sin\theta$ is positive means that the terminal side is in QI. We draw a reference triangle in QI with $r = 3$ and $x = 1$ (Figure 4.34 on the next page); then we use the Pythagorean Theorem to find that $y = \sqrt{3^2 - 1^2} = \sqrt{8}$. (Note that y is positive in QI.)

We then use the definitions to get

$$\cos\theta = \frac{1}{3} \quad \text{and} \quad \tan\theta = \frac{\sqrt{8}}{1}.$$

(We could also have found $\cos\theta$ directly as the reciprocal of $\sec\theta$.)

(c) Since $\cot\theta$ is undefined, we conclude that $y = 0$ and that θ is a quadrantal angle on the x-axis. The added fact that $\sec\theta$ is negative means that the terminal side is along the negative x-axis. We choose the point $(-1, 0)$ on the terminal side and use the definitions to get

$$\cos\theta = -1 \quad \text{and} \quad \tan\theta = \frac{0}{-1} = 0.$$

Now try Exercise 43.

Why Not Degrees?

One could actually develop a consistent theory of trigonometric functions based on a rescaled x-axis with "degrees." For example, your graphing calculator will probably produce reasonable-looking graphs in degree mode. Calculus, however, uses rules that *depend* on radian measure for all trigonometric functions, so it is prudent for precalculus students to become accustomed to that now.

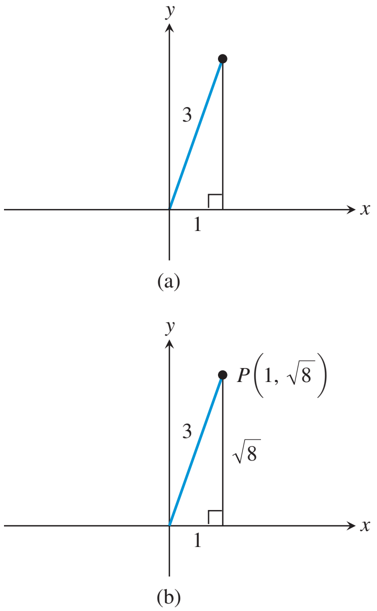


FIGURE 4.34 (Example 7b)

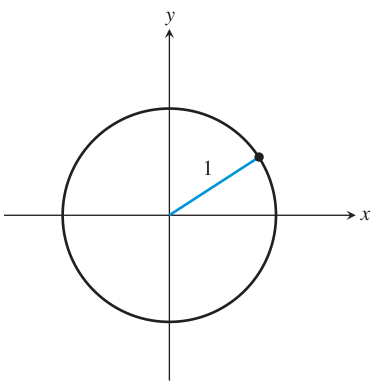


FIGURE 4.35 The unit circle.

Trigonometric Functions of Real Numbers

Now that we have extended the six basic trigonometric functions to apply to any angle, we are ready to appreciate them as functions of real numbers and to study their behavior. First, for reasons discussed in the first section of this chapter, we must agree to measure θ in radian mode so that the real number units of the input will match the real number units of the output.

When considering the trigonometric functions as functions of real numbers, the angles will be measured in radians.

DEFINITION Unit Circle

The unit circle is a circle of radius 1 centered at the origin (Figure 4.35).

The unit circle provides an ideal connection between triangle trigonometry and the trigonometric functions. Because arc length along the unit circle corresponds exactly to radian measure, we can use the circle itself as a sort of "number line" for the input values of our functions. This involves the **wrapping function**, which associates points on the number line with points on the circle.

Figure 4.36 shows how the wrapping function works. The real line is placed tangent to the unit circle at the point $(1, 0)$, the point from which we measure angles in standard position. When the line is wrapped around the unit circle in both the positive (counterclockwise) and negative (clockwise) directions, each point t on the real line will fall on a point of the circle that lies on the terminal side of an angle of t radians in standard position. Using the coordinates (x, y) of this point, we can find the six trigonometric ratios for the angle t just as we did in Example 7—except even more easily, since $r = 1$.

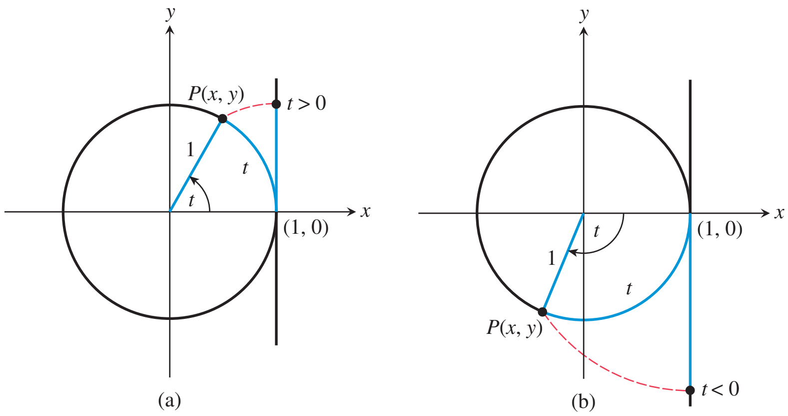


FIGURE 4.36 How the number line is wrapped onto the unit circle. Note that each number t (positive or negative) is "wrapped" to a point P that lies on the terminal side of an angle of t radians in standard position.

DEFINITION Trigonometric Functions of Real Numbers

Let t be any real number, and let $P(x, y)$ be the point corresponding to t when the number line is wrapped onto the unit circle as described above. Then

$$\sin t = y \qquad \csc t = \frac{1}{y}\ (y \neq 0)$$

$$\cos t = x \qquad \sec t = \frac{1}{x}\ (x \neq 0)$$

$$\tan t = \frac{y}{x}\ (x \neq 0) \qquad \cot t = \frac{x}{y}\ (y \neq 0)$$

Therefore, the number t on the number line always wraps onto the point $(\cos t, \sin t)$ on the unit circle (Figure 4.37).

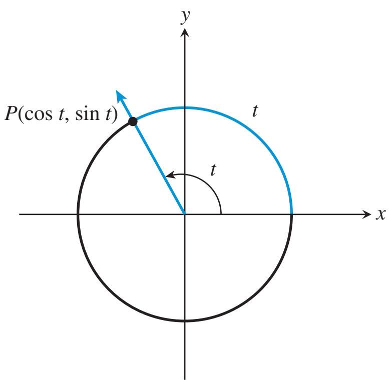


FIGURE 4.37 The real number t always wraps onto the point $(\cos t, \sin t)$ on the unit circle.

Although it is still helpful to draw reference triangles inside the unit circle to see the ratios geometrically, this latest round of definitions does not invoke triangles at all. The real number t determines a point on the unit circle, and the (x, y) coordinates of the point determine the six trigonometric ratios. For this reason, the trigonometric functions when applied to real numbers are usually called the **circular functions**.

EXPLORATION 2 Exploring the Unit Circle

This works well as a group exploration. Get together in groups of two or three and explain to each other why these statements are true. Base your explanations on the unit circle (Figure 4.37). Remember that $-t$ wraps the same distance as t, but in the opposite direction.

1. For any t, the value of $\cos t$ lies between -1 and 1 inclusive.
2. For any t, the value of $\sin t$ lies between -1 and 1 inclusive.
3. The values of $\cos t$ and $\cos(-t)$ are always equal to each other. (Recall that this is the check for an *even* function.)
4. The values of $\sin t$ and $\sin(-t)$ are always opposites of each other. (Recall that this is the check for an *odd* function.)
5. The values of $\sin t$ and $\sin(t + 2\pi)$ are always equal to each other. In fact, that is true of all six trig functions on their domains, and for the same reason.
6. The values of $\sin t$ and $\sin(t + \pi)$ are always opposites of each other. The same is true of $\cos t$ and $\cos(t + \pi)$.
7. The values of $\tan t$ and $\tan(t + \pi)$ are always equal to each other (unless they are both undefined).
8. The sum $(\cos t)^2 + (\sin t)^2$ always equals 1.
9. (Challenge) Can you discover a similar relationship that is not mentioned in our list of eight? There are some to be found.

Periodic Functions

Statements 5 and 7 in Exploration 2 reveal an important property of the circular functions that we need to define for future reference.

DEFINITION Periodic Function

A function $y = f(t)$ is **periodic** if there is a positive number c such that $f(t + c) = f(t)$ for all values of t in the domain of f. The smallest such number c is called the **period** of the function.

Exploration 2 suggests that the sine and cosine functions have period 2π and that the tangent function has period π. We use this periodicity later to model predictably repetitive behavior in the real world, but meanwhile we can also use it to solve little noncalculator training problems like in some of the previous examples in this section.

EXAMPLE 8 Using Periodicity

Find each of the following numbers without a calculator.

(a) $\sin\left(\dfrac{57{,}801\pi}{2}\right)$ **(b)** $\cos(288.45\pi) - \cos(280.45\pi)$

(c) $\tan\left(\dfrac{\pi}{4} - 99{,}999\pi\right)$

(continued)

SOLUTION

(a) $$\sin\left(\frac{57{,}801\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \frac{57{,}800\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 28{,}900\pi\right)$$
$$= \sin\left(\frac{\pi}{2}\right) = 1$$

Notice that $28{,}900\pi$ is just a large multiple of 2π, so $\pi/2$ and $((\pi/2) + 28{,}900\pi)$ wrap to the same point on the unit circle, namely $(0, 1)$.

(b) $\cos(288.45\pi) - \cos(280.45\pi) =$
$\cos(280.45\pi + 8\pi) - \cos(280.45\pi) = 0$

Notice that 280.45π and $(280.45\pi + 8\pi)$ wrap to the same point on the unit circle, so the cosine of one is the same as the cosine of the other.

(c) Since the period of the tangent function is π rather than 2π, $99{,}999\pi$ is a large multiple of the period of the tangent function. Therefore,

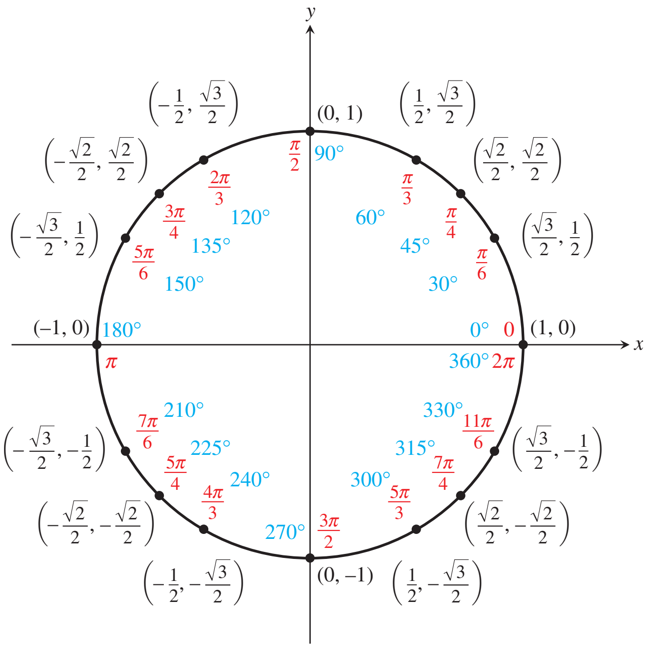
$$\tan\left(\frac{\pi}{4} - 99{,}999\pi\right) = \tan\left(\frac{\pi}{4}\right) = 1.$$

Now try Exercise 49.

We take a closer look at the properties of the six circular functions in the next two sections.

The 16-Point Unit Circle

At this point you should be able to use reference triangles and quadrantal angles to evaluate trigonometric functions for all integer multiples of 30° or 45° (equivalently, $\pi/6$ radians or $\pi/4$ radians). All of these special values wrap to the 16 special points shown on the unit circle below. Study this diagram until you are confident that you can find the coordinates of these points easily, but avoid using it as a reference when doing problems.



QUICK REVIEW 4.3 *(For help, go to Section 4.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, give the value of the angle θ in degrees.

1. $\theta = -\dfrac{\pi}{6}$ **2.** $\theta = -\dfrac{5\pi}{6}$

3. $\theta = \dfrac{25\pi}{4}$ **4.** $\theta = \dfrac{16\pi}{3}$

In Exercises 5–8, use special triangles to evaluate:

5. $\tan \dfrac{\pi}{6}$ **6.** $\cot \dfrac{\pi}{4}$

7. $\csc \dfrac{\pi}{4}$ **8.** $\sec \dfrac{\pi}{3}$

In Exercises 9 and 10, use a right triangle to find the other five trigonometric functions of the *acute* angle θ.

9. $\sin \theta = \dfrac{5}{13}$ **10.** $\cos \theta = \dfrac{15}{17}$

SECTION 4.3 EXERCISES

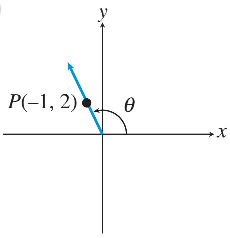
In Exercises 1 and 2, identify the one angle that is not coterminal with all the others.

1. 150°, 510°, −210°, 450°, 870°

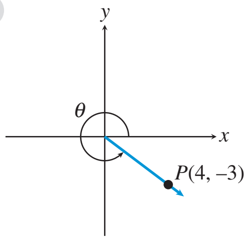
2. $\dfrac{5\pi}{3}, -\dfrac{5\pi}{3}, \dfrac{11\pi}{3}, -\dfrac{7\pi}{3}, \dfrac{365\pi}{3}$

In Exercises 3–6, evaluate the six trigonometric functions of the angle θ.

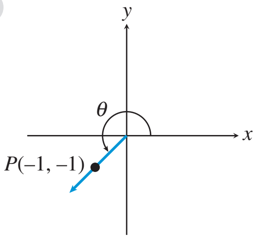
3.



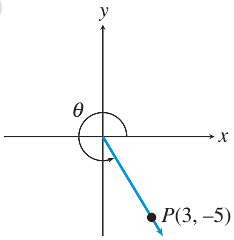
4.



5.



6.



In Exercises 7–12, point P is on the terminal side of angle θ. Evaluate the six trigonometric functions for θ. If the function is undefined, write "undefined."

7. $P(3, 4)$ **8.** $P(-4, -6)$

9. $P(0, 5)$ **10.** $P(-3, 0)$

11. $P(5, -2)$ **12.** $P(22, -22)$

In Exercises 13–16, state the sign $(+ \text{ or } -)$ of **(a)** $\sin t$, **(b)** $\cos t$, and **(c)** $\tan t$ for values of t in the interval given.

13. $\left(0, \dfrac{\pi}{2}\right)$ **14.** $\left(\dfrac{\pi}{2}, \pi\right)$

15. $\left(\pi, \dfrac{3\pi}{2}\right)$ **16.** $\left(\dfrac{3\pi}{2}, 2\pi\right)$

In Exercises 17–20, determine the sign $(+ \text{ or } -)$ of the given value without the use of a calculator.

17. $\cos 143°$ **18.** $\tan 192°$

19. $\cos \dfrac{7\pi}{8}$ **20.** $\tan \dfrac{4\pi}{5}$

In Exercises 21–24, choose the point on the terminal side of θ.

21. $\theta = 45°$

(a) $(2, 2)$ **(b)** $(1, \sqrt{3})$ **(c)** $(\sqrt{3}, 1)$

22. $\theta = \dfrac{2\pi}{3}$

(a) $(-1, 1)$ **(b)** $(-1, \sqrt{3})$ **(c)** $(-\sqrt{3}, 1)$

23. $\theta = \dfrac{7\pi}{6}$

(a) $(-\sqrt{3}, -1)$ **(b)** $(-1, \sqrt{3})$ **(c)** $(-\sqrt{3}, 1)$

24. $\theta = -60°$

(a) $(-1, -1)$ **(b)** $(1, -\sqrt{3})$ **(c)** $(-\sqrt{3}, 1)$

In Exercises 25–36, evaluate without using a calculator by using ratios in a reference triangle.

25. $\cos 120°$ **26.** $\tan 300°$

27. $\sec \dfrac{\pi}{3}$ **28.** $\csc \dfrac{3\pi}{4}$

29. $\sin \dfrac{13\pi}{6}$ **30.** $\cos \dfrac{7\pi}{3}$

31. $\tan -\dfrac{15\pi}{4}$ **32.** $\cot \dfrac{13\pi}{4}$

33. $\cos \dfrac{23\pi}{6}$ **34.** $\cos \dfrac{17\pi}{4}$

35. $\sin \dfrac{11\pi}{3}$ **36.** $\cot \dfrac{19\pi}{6}$

In Exercises 37–42, find **(a)** $\sin \theta$, **(b)** $\cos \theta$, and **(c)** $\tan \theta$ for the given quadrantal angle. If the value is undefined, write "undefined."

37. $-450°$ **38.** $-270°$

39. 7π **40.** $\dfrac{11\pi}{2}$

41. $-\dfrac{7\pi}{2}$ **42.** -4π

In Exercises 43–48, evaluate without using a calculator.

43. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{2}{3}$ and $\cot\theta > 0$.

44. Find $\cos\theta$ and $\cot\theta$ if $\sin\theta = \frac{1}{4}$ and $\tan\theta < 0$.

45. Find $\tan\theta$ and $\sec\theta$ if $\sin\theta = -\frac{2}{5}$ and $\cos\theta > 0$.

46. Find $\sin\theta$ and $\cos\theta$ if $\cot\theta = \frac{3}{7}$ and $\sec\theta < 0$.

47. Find $\sec\theta$ and $\csc\theta$ if $\cot\theta = -\frac{4}{3}$ and $\cos\theta < 0$.

48. Find $\csc\theta$ and $\cot\theta$ if $\tan\theta = -\frac{4}{3}$ and $\sin\theta > 0$.

In Exercises 49–52, evaluate by using the period of the function.

49. $\sin\left(\frac{\pi}{6} + 49{,}000\pi\right)$

50. $\tan\,(1{,}234{,}567\pi) - \tan\,(7{,}654{,}321\pi)$

51. $\cos\left(\frac{5{,}555{,}555\pi}{2}\right)$

52. $\tan\left(\frac{3\pi - 70{,}000\pi}{2}\right)$

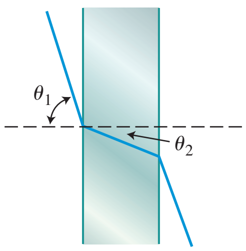
53. Group Activity Use a calculator to evaluate the expressions in Exercises 49–52. Does your calculator give the correct answers? Many calculators miss all four. Give a brief explanation of what probably goes wrong.

54. Writing to Learn Give a convincing argument that the period of $\sin t$ is 2π. That is, show that there is no smaller positive real number p such that $\sin\,(t + p) = \sin t$ for all real numbers t.

55. Refracted Light Light is *refracted* (bent) as it passes through glass. In the figure, θ_1 is the angle of incidence and θ_2 is the *angle of refraction*. The *index of refraction* is a constant μ that satisfies the equation

$$\sin\theta_1 = \mu\sin\theta_2.$$

If $\theta_1 = 83°$ and $\theta_2 = 36°$ for a certain piece of flint glass, find the index of refraction.



Glass

56. Refracted Light A certain piece of crown glass has an index of refraction of 1.52. If a light ray enters the glass at an angle $\theta_1 = 42°$, what is $\sin\theta_2$?

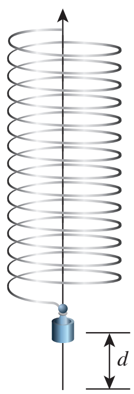
57. Damped Harmonic Motion A weight suspended from a spring is set into motion. Its displacement d from equilibrium is modeled by the equation

$$d = 0.4e^{-0.2t}\cos 4t,$$

where d is the displacement in inches and t is the time in seconds. Find the displacement at the given time. Use radian mode.

(a) $t = 0$

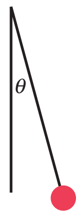
(b) $t = 3$



58. Swinging Pendulum The Columbus Museum of Science and Industry exhibits a Foucault pendulum 32 ft long that swings back and forth on a cable once in approximately 6 sec. The angle θ (in radians) between the cable and an imaginary vertical line is modeled by the equation

$$\theta = 0.25\cos t.$$

Find the measure of angle θ when $t = 0$ and $t = 2.5$.



59. Too Close for Comfort An F-15 aircraft flying at an altitude of 8000 ft passes directly over a group of vacationers hiking at 7400 ft. If θ is the angle of elevation from the hikers to the F-15, find the distance d from the group to the jet for the given angle.

(a) $\theta = 45°$ **(b)** $\theta = 90°$ **(c)** $\theta = 140°$

60. Manufacturing Swimwear Get Wet, Inc., manufactures swimwear, a seasonal product. The monthly sales x (in thousands) for Get Wet swimsuits are modeled by the equation

$$x = 72.4 + 61.7\sin\frac{\pi t}{6},$$

where $t = 1$ represents January, $t = 2$ February, and so on. Estimate the number of Get Wet swimsuits sold in January, April, June, October, and December. For which two of these months are sales the same? Explain why this might be so.

Standardized Test Questions

61. True or False If θ is an angle of a triangle such that $\cos\theta < 0$, then θ is obtuse. Justify your answer.

62. True or False If θ is an angle in standard position determined by the point $(8, -6)$, then $\sin\theta = -0.6$. Justify your answer.

You should answer these questions without using a calculator.

63. Multiple Choice If $\sin\theta = 0.4$, then $\sin(-\theta) + \csc\theta =$

(A) -0.15. **(B)** 0. **(C)** 0.15. **(D)** 0.65. **(E)** 2.1.

64. Multiple Choice If $\cos\theta = 0.4$, then $\cos\,(\theta + \pi) =$

(A) -0.6. **(B)** -0.4. **(C)** 0.4. **(D)** 0.6. **(E)** 3.54.

65. Multiple Choice The range of the function $f(t) = (\sin t)^2 + (\cos t)^2$ is

(A) $\{1\}$. **(B)** $[-1, 1]$. **(C)** $[0, 1]$.

(D) $[0, 2]$. **(E)** $[0, \infty)$.

66. Multiple Choice If $\cos\theta = -\frac{5}{13}$ and $\tan\theta > 0$, then $\sin\theta =$

(A) $-\frac{12}{13}$. **(B)** $-\frac{5}{12}$. **(C)** $\frac{5}{13}$. **(D)** $\frac{5}{12}$. **(E)** $\frac{12}{13}$.

Explorations

In Exercises 67–70, find the value of the unique real number θ between 0 and 2π that satisfies the two given conditions.

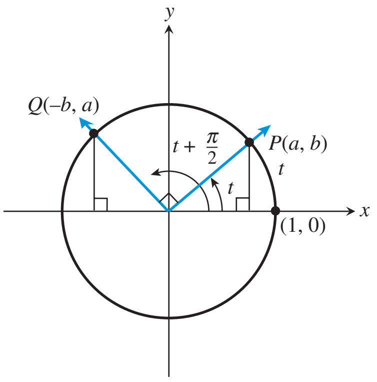
67. $\sin\theta = \frac{1}{2}$ and $\tan\theta < 0$.

68. $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta < 0$.

69. $\tan\theta = -1$ and $\sin\theta < 0$.

70. $\sin\theta = -\dfrac{\sqrt{2}}{2}$ and $\tan\theta > 0$.

Exercises 71–74 refer to the unit circle in this figure. Point P is on the terminal side of an angle t and point Q is on the terminal side of an angle $t + \pi/2$.



71. Using Geometry in Trigonometry Drop perpendiculars from points P and Q to the x-axis to form two right triangles. Explain how the right triangles are related.

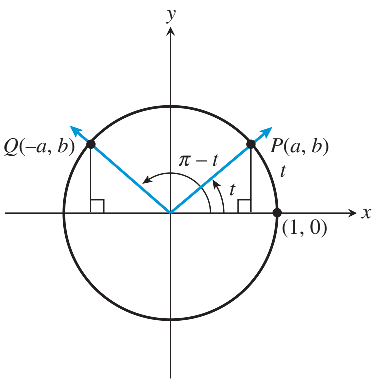
72. Using Geometry in Trigonometry If the coordinates of point P are (a, b), explain why the coordinates of point Q are $(-b, a)$.

73. Explain why $\sin\left(t + \dfrac{\pi}{2}\right) = \cos t$.

74. Explain why $\cos\left(t + \dfrac{\pi}{2}\right) = -\sin t$.

75. Writing to Learn In the figure for Exercises 71–74, t is an angle with radian measure $0 < t < \pi/2$. Draw a similar figure for an angle with radian measure $\pi/2 < t < \pi$ and use it to explain why $\sin(t + \pi/2) = \cos t$.

76. Writing to Learn Use the accompanying figure to explain each of the following.



(a) $\sin(\pi - t) = \sin t$ **(b)** $\cos(\pi - t) = -\cos t$

Extending the Ideas

77. Approximation and Error Analysis Use your grapher to complete the table to show that $\sin\theta \approx \theta$ (in radians) when $|\theta|$ is small. Physicists often use the approximation $\sin\theta \approx \theta$ for small values of θ. For what values of θ is the *magnitude of the error* in approximating $\sin\theta$ by θ less than 1% of $\sin\theta$? That is, solve the relation

$$|\sin\theta - \theta| < 0.01|\sin\theta|.$$

[*Hint:* Extend the table to include a column for values of

$$\frac{|\sin\theta - \theta|}{|\sin\theta|}.]$$

θ	$\sin\theta$	$\sin\theta - \theta$
-0.03		
-0.02		
-0.01		
0		
0.01		
0.02		
0.03		

78. Proving a Theorem If t is any real number, prove that $1 + (\tan t)^2 = (\sec t)^2$.

Taylor Polynomials Radian measure allows the trigonometric functions to be approximated by simple polynomial functions. For example, in Exercises 79 and 80, sine and cosine are approximated by Taylor polynomials, named after the English mathematician Brook Taylor (1685–1731). Complete each table showing a Taylor polynomial in the third column. Describe the patterns in the table.

79.

θ	$\sin\theta$	$\theta - \dfrac{\theta^3}{6}$	$\sin\theta - \left(\theta - \dfrac{\theta^3}{6}\right)$
-0.3	$-0.295\ldots$		
-0.2	$-0.198\ldots$		
-0.1	$-0.099\ldots$		
0	0		
0.1	$0.099\ldots$		
0.2	$0.198\ldots$		
0.3	$0.295\ldots$		

80.

θ	$\cos\theta$	$1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}$	$\cos\theta - \left(1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}\right)$
-0.3	$0.955\ldots$		
-0.2	$0.980\ldots$		
-0.1	$0.995\ldots$		
0	1		
0.1	$0.995\ldots$		
0.2	$0.980\ldots$		
0.3	$0.955\ldots$		