3.5 Equation Solving and Modeling

What you'll learn about

- Solving Exponential Equations
- Solving Logarithmic Equations
- Orders of Magnitude and Logarithmic Models
- Newton's Law of Cooling
- Logarithmic Re-expression

... and why

The Richter scale, pH, and Newton's Law of Cooling are among the most important uses of logarithmic and exponential functions.

Solving Exponential Equations

Some logarithmic equations can be solved by changing to exponential form, as we saw in Example 5 of Section 3.3. For other equations, the properties of exponents or the properties of logarithms are used. A property of both exponential and logarithmic functions that is often helpful for solving equations is that they are one-to-one functions.

One-to-One Properties

For any exponential function $f(x) = b^x$,

- If $b^u = b^v$, then $u = v$.

For any logarithmic function $f(x) = \log_b x$,

- If $\log_b u = \log_b v$, then $u = v$.

Example 1 shows how the one-to-oneness of exponential functions can be used. Examples 3 and 4 use the one-to-one property of logarithms.

EXAMPLE 1 Solving an Exponential Equation Algebraically

Solve $20(1/2)^{x/3} = 5$.

SOLUTION

$$20\left(\frac{1}{2}\right)^{x/3} = 5$$

$$\left(\frac{1}{2}\right)^{x/3} = \frac{1}{4} \qquad \text{Divide by 20.}$$

$$\left(\frac{1}{2}\right)^{x/3} = \left(\frac{1}{2}\right)^2 \qquad \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

$$\frac{x}{3} = 2 \qquad \text{One-to-one property}$$

$$x = 6 \qquad \text{Multiply by 3.}$$

Now try Exercise 1.

The equation in Example 2 involves a difference of two exponential functions, which makes it difficult to solve algebraically. So we start with a graphical approach.

EXAMPLE 2 Solving an Exponential Equation

Solve $(e^x - e^{-x})/2 = 5$.

SOLUTION

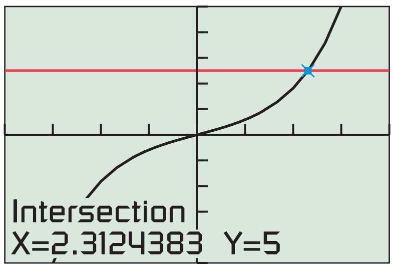
Solve Graphically Figure 3.32 shows that the graphs of $y = (e^x - e^{-x})/2$ and $y = 5$ intersect when $x \approx 2.31$.

Confirm Algebraically The algebraic approach involves some ingenuity. If we multiply each side of the original equation by $2e^x$ and rearrange the terms, we can obtain a quadratic equation in e^x:

$$\frac{e^x - e^{-x}}{2} = 5$$

$$e^{2x} - e^0 = 10e^x \qquad \text{Multiply by } 2e^x.$$

$$(e^x)^2 - 10(e^x) - 1 = 0 \qquad \text{Subtract } 10e^x.$$



[–4, 4] by [–10, 10]

FIGURE 3.32 $y = (e^x - e^{-x})/2$ and $y = 5$. (Example 2)

A Cinch?

You may recognize the left-hand side of the equation in Example 2 as the *hyperbolic sine function* that was introduced in Exercise 59 of Section 3.2. This function is often used in calculus. We write $\sinh(x) = (e^x - e^{-x})/2$. "Sinh" is pronounced as if spelled "cinch."

If we let $w = e^x$, this equation becomes $w^2 - 10w - 1 = 0$, and the quadratic formula gives

$$w = e^x = \frac{10 \pm \sqrt{104}}{2} = 5 \pm \sqrt{26}.$$

Because e^x is always positive, we reject the possibility that e^x has the negative value $5 - \sqrt{26}$. Therefore,

$$e^x = 5 + \sqrt{26}$$

$$x = \ln(5 + \sqrt{26}) \qquad \text{Convert to logarithmic form.}$$

$$x = 2.312\ldots \approx 2.31 \qquad \text{Approximate with a grapher.}$$

Now try Exercise 31.

Solving Logarithmic Equations

When logarithmic equations are solved algebraically, it is important to keep track of the domain of each expression in the equation as it is being solved. A particular algebraic method may introduce extraneous solutions, or worse yet, *lose* some valid solutions, as illustrated in Example 3.

EXAMPLE 3 Solving a Logarithmic Equation

Solve $\log x^2 = 2$.

SOLUTION

Method 1 Use the one-to-one property of logarithms.

$$\log x^2 = 2$$

$$\log x^2 = \log 10^2 \qquad y = \log 10^y$$

$$x^2 = 10^2 \qquad \text{One-to-one property}$$

$$x^2 = 100 \qquad 10^2 = 100$$

$$x = 10 \quad \text{or} \quad x = -10$$

Method 2 Change the equation from logarithmic to exponential form.

$$\log x^2 = 2$$

$$x^2 = 10^2 \qquad \text{Change to exponential form.}$$

$$x^2 = 100 \qquad 10^2 = 100$$

$$x = 10 \quad \text{or} \quad x = -10$$

Method 3 (Incorrect) Use the power rule of logarithms.

$$\log x^2 = 2$$

$$2\log x = 2 \qquad \text{Power rule, incorrectly applied}$$

$$\log x = 1 \qquad \text{Divide by 2.}$$

$$x = 10 \qquad \text{Change to exponential form.}$$

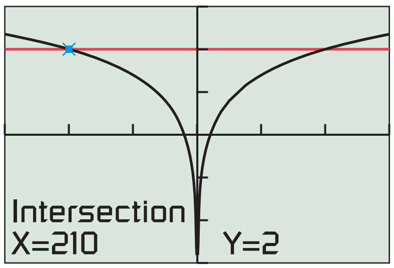
Support Graphically

Figure 3.33 shows that the graphs of $f(x) = \log x^2$ and $y = 2$ intersect when $x = -10$. From the symmetry of the graphs due to f being an even function, we can see that $x = 10$ is also a solution.

Interpret

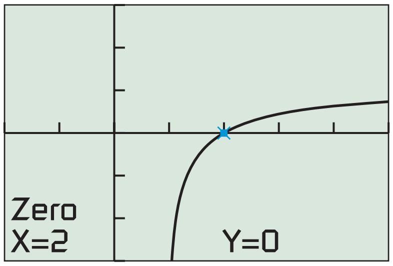
Methods 1 and 2 are correct. Method 3 fails because the domain of $\log x^2$ is all nonzero real numbers, but the domain of $\log x$ is only the positive real numbers. The correct solution includes both 10 and -10 because both of these x-values make the original equation true.

Now try Exercise 25.



[–15, 15] by [–3, 3]

FIGURE 3.33 Graphs of $f(x) = \log x^2$ and $y = 2$. (Example 3)



[–2, 5] by [–3, 3]

FIGURE 3.34 The zero of $f(x) = \ln(3x - 2) + \ln(x - 1) - 2\ln x$ is $x = 2$. (Example 4)

Method 3 above violates an easily overlooked condition of the power rule $\log_b R^c = c \log_a R$, namely, that the rule holds *if* R is *positive*. In the expression $\log x^2$, x plays the role of R, and x can be -10, which is *not* positive. Because algebraic manipulation of a logarithmic equation can produce expressions with different domains, a graphical solution is often less prone to error.

EXAMPLE 4 Solving a Logarithmic Equation

Solve $\ln(3x - 2) + \ln(x - 1) = 2\ln x$.

SOLUTION To use the x-intercept method, we rewrite the equation as

$$\ln(3x - 2) + \ln(x - 1) - 2\ln x = 0,$$

and graph

$$f(x) = \ln(3x - 2) + \ln(x - 1) - 2\ln x,$$

as shown in Figure 3.34. The x-intercept is $x = 2$, which is the solution to the equation.

Now try Exercise 35.

Orders of Magnitude and Logarithmic Models

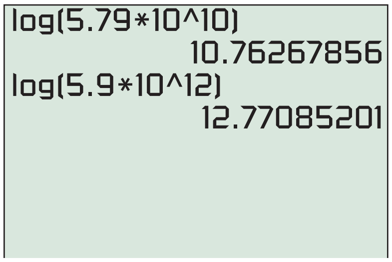


FIGURE 3.35 Pluto is two orders of magnitude farther from the Sun than Mercury.

When comparing quantities, their sizes sometimes span a wide range. This is why scientific notation was developed.

For instance, the planet Mercury is 57.9 billion meters from the Sun; whereas Pluto is 5900 billion meters from the Sun, roughly 100 times farther! In scientific notation, Mercury is 5.79×10^{10} m from the Sun, and Pluto is 5.9×10^{12} m from the Sun. Pluto's distance is 2 powers of ten greater than Mercury's distance. From Figure 3.35, we see that the difference in the common logs of these two distances is about 2. The common logarithm of a positive quantity is its **order of magnitude**. So we say, Pluto's distance from the Sun is 2 orders of magnitude greater than Mercury's.

Orders of magnitude can be used to compare any like quantities:

- A kilometer is 3 orders of magnitude longer than a meter.
- A dollar is 2 orders of magnitude greater than a penny.
- A horse weighing 400 kg is 4 orders of magnitude heavier than a mouse weighing 40 g.
- New York City with 8 million people is 6 orders of magnitude bigger than Earmuff Junction with a population of 8.

EXPLORATION 1 Comparing Scientific Notation and Common Logarithms

1. Using a calculator compute $\log(4 \cdot 10)$, $\log(4 \cdot 10^2)$, $\log(4 \cdot 10^3), \ldots,$ $\log(4 \cdot 10^{10})$.
2. What is the pattern in the integer parts of these numbers?
3. What is the pattern of their decimal parts?
4. How many orders of magnitude greater is $4 \cdot 10^{10}$ than $4 \cdot 10$?

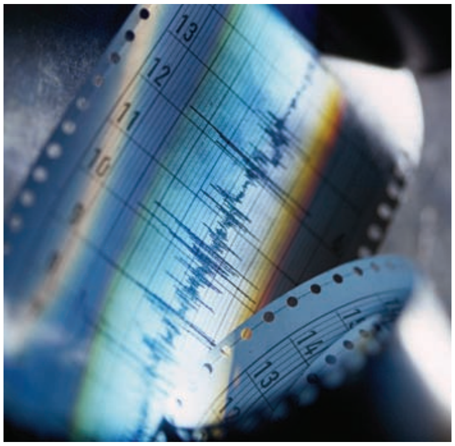
Orders of magnitude have many applications. For a sound or noise, the *bel*, mentioned in Section 3.3, measures the order of magnitude of its intensity compared to the threshold of hearing. For instance, a sound of 3 bels or 30 dB (decibels) has a sound intensity 3 orders of magnitude above the threshold of hearing.

Orders of magnitude are also used to compare the severity of earthquakes and the acidity of chemical solutions. We now turn our attention to these two applications.

As mentioned in Exercise 52 of Section 3.4, the *Richter scale* magnitude R of an earthquake is

$$R = \log \frac{a}{T} + B,$$

where a is the amplitude in micrometers (μm) of the vertical ground motion at the receiving station, T is the period of the associated seismic wave in seconds, and B accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.



EXAMPLE 5 Comparing Earthquake Intensities

How many times more severe was the 2001 earthquake in Gujarat, India $(R_1 = 7.9)$ than the 1999 earthquake in Athens, Greece $(R_2 = 5.9)$?

SOLUTION

Model

The severity of an earthquake is measured by the associated amplitude. Let a_1 be the amplitude for the Gujarat earthquake and a_2 be the amplitude for the Athens earthquake. Then

$$R_1 = \log \frac{a_1}{T} + \mathrm{B} = 7.9$$

$$R_2 = \log \frac{a_2}{T} + B = 5.9$$

Solve Algebraically

We seek the ratio of severities a_1/a_2:

$$\left(\log \frac{a_1}{T} + B\right) - \left(\log \frac{a_2}{T} + B\right) = R_1 - R_2$$

$$\log \frac{a_1}{T} - \log \frac{a_2}{T} = 7.9 - 5.9 \qquad B - B = 0$$

$$\log \frac{a_1}{a_2} = 2 \qquad \text{Quotient rule}$$

$$\frac{a_1}{a_2} = 10^2 = 100$$

Interpret

A Richter scale difference of 2 corresponds to an amplitude ratio of 2 powers of 10, or $10^2 = 100$. So the Gujarat quake was 100 times as severe as the Athens quake.

Now try Exercise 45.

In chemistry, the acidity of a water-based solution is measured by the concentration of hydrogen ions in the solution (in moles per liter). The hydrogen-ion concentration is written $[\mathrm{H}^+]$. Because such concentrations usually involve *negative* powers of ten, *negative* orders of magnitude are used to compare acidity levels. The measure of acidity used is **pH**, the opposite of the common log of the hydrogen-ion concentration:

$$\mathrm{pH} = -\log\,[\mathrm{H}^+]$$

More acidic solutions have higher hydrogen-ion concentrations and lower pH values.

EXAMPLE 6 Comparing Chemical Acidity

Some especially sour vinegar has a pH of 2.4, and a box of Leg and Sickle baking soda has a pH of 8.4.

(a) What are their hydrogen-ion concentrations?

(b) How many times greater is the hydrogen-ion concentration of the vinegar than that of the baking soda?

(c) By how many orders of magnitude do the concentrations differ?

SOLUTION

(a)

$$\text{Vinegar: } -\log\,[H^+] = 2.4$$
$$\log\,[H^+] = -2.4$$
$$[H^+] = 10^{-2.4} \approx 3.98 \times 10^{-3} \text{ moles per liter}$$

$$\text{Baking soda: } -\log\,[H^+] = 8.4$$
$$\log\,[H^+] = -8.4$$
$$[H^+] = 10^{-8.4} \approx 3.98 \times 10^{-9} \text{ moles per liter}$$

(b)

$$\frac{[H^+] \text{ of vinegar}}{[H^+] \text{ of baking soda}} = \frac{10^{-2.4}}{10^{-8.4}} = 10^{(-2.4)-(-8.4)} = 10^6$$

(c) The hydrogen-ion concentration of the vinegar is 6 orders of magnitude greater than that of the Leg and Sickle baking soda, exactly the difference in their pH values. *Now try Exercise 47.*

Newton's Law of Cooling

An object that has been heated will cool to the temperature of the medium in which it is placed, such as the surrounding air or water. The temperature T of the object at time t can be modeled by

$$T(t) = T_m + (t_0 - T_m)e^{-kt}$$

for an appropriate value of k, where

$$T_m = \text{the temperature of the surrounding medium,}$$
$$T_0 = \text{initial temperature of the object.}$$

This model assumes that the surrounding medium, although taking heat from the object, essentially maintains a constant temperature. In honor of English mathematician and physicist Isaac Newton (1643–1727), this model is called **Newton's Law of Cooling**.

EXAMPLE 7 Applying Newton's Law of Cooling

A hard-boiled egg at temperature 96°C is placed in 16°C water to cool. Four minutes later the temperature of the egg is 45°C. Use Newton's Law of Cooling to determine when the egg will be 20°C.

SOLUTION

Model Because $T_0 = 96$ and $T_m = 16$, $T_0 - T_m = 80$ and

$$T(t) = T_m + (T_0 - T_m)e^{-kt} = 16 + 80e^{-kt}.$$

To find the value of k we use the fact that $T = 45$ when $t = 4$.

$$45 = 16 + 80e^{-4k}$$

$$\frac{29}{80} = e^{-4k} \qquad \text{Subtract 16, then divide by 80.}$$

$$\ln\frac{29}{80} = -4k \qquad \text{Change to logarithmic form.}$$

$$k = -\frac{\ln(29/80)}{4} \qquad \text{Divide by } -4.$$

$$k = 0.253\ldots$$

We save this k-value because it is part of our model. (See Figure 3.36.)

```
-ln(29/80)/4 → K
          .2536827012
-ln(1/20)/K
          11.80897341
```

FIGURE 3.36 Storing and using the constant k.

Solve Algebraically To find t when $T = 20°C$, we solve the equation:

$$20 = 16 + 80e^{-kt}$$

$$\frac{4}{80} = e^{-kt} \qquad \text{Subtract 16, then divide by 80.}$$

$$\ln\frac{4}{80} = -kt \qquad \text{Change to logarithmic form.}$$

$$t = -\frac{\ln(4/80)}{k} \approx 11.81 \qquad \text{See Figure 3.36.}$$

Interpret The temperature of the egg will be 20°C after about 11.81 min (11 min 49 sec).

Now try Exercise 49.

We can rewrite Newton's Law of Cooling in the following form:

$$T(t) - T_m = (T_0 - T_m)e^{-kt}$$

We use this form of Newton's Law of Cooling when modeling temperature using data gathered from an actual experiment. Because the difference $T - T_m$ is an exponential function of time t, we can use exponential regression on $T - T_m$ versus t to obtain a model, as illustrated in Example 8.

EXAMPLE 8 Modeling with Newton's Law of Cooling

In an experiment, a temperature probe connected to a Calculator-Based-Laboratory™ device was removed from a cup of hot coffee and placed in a glass of cold water. The first two columns of Table 3.23 show the resulting data for time t (in seconds since the probe was placed in the water) and temperature T (in °C). In the third column, the temperature data have been *re-expressed* by subtracting the temperature of the water, which was 4.5°C.

(a) Estimate the temperature of the coffee.

(b) Estimate the time when the temperature probe reading was 40°C.

Table 3.23 Temperature Data from a CBL™ Experiment

Time t	Temp T	$T - T_m$
2	64.8	60.3
5	49.0	44.5
10	31.4	26.9
15	22.0	17.5
20	16.5	12.0
25	14.2	9.7
30	12.0	7.5

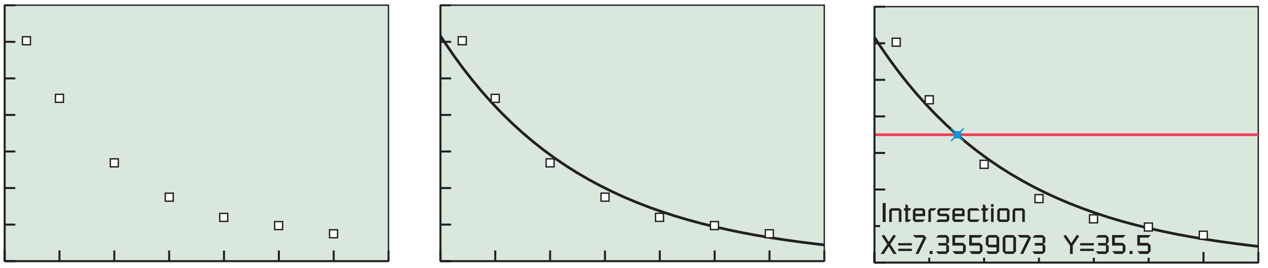
SOLUTION

Model Figure 3.37a shows a scatter plot of the re-expressed temperature data. Using exponential regression, we obtain the following model:

$$T(t) - 4.5 = 61.656 \times 0.92770^t$$

Figure 3.37b shows the graph of this model with the scatter plot of the data. You can see that the curve fits the data fairly well.

(continued)



[0, 35] by [0, 70]
(a)

[0, 35] by [0, 70]
(b)

[0, 35] by [0, 70]
(c)

FIGURE 3.37 Scatter plot and graphs for Example 8.

(a) Solve Algebraically From the model we see that $T_0 - T_m \approx 61.656$. So

$$T_0 \approx 61.656 + T_m = 61.656 + 4.5 \approx 66.16$$

(b) Solve Graphically Figure 3.37c shows that the graph of $T(t) - 4.5 = 61.656 \times 0.92770^t$ intersects $y = 40 - 4.5 = 35.5$ when $t \approx 7.36$.

Interpret The temperature of the coffee was roughly 66.2°C, and the probe reading was 40°C about 7.4 sec after it was placed in the water. ***Now try Exercise 51.***

Logarithmic Re-expression

In Example 7 of Section 3.4 we learned that data pairs (x, y) that fit a power model have a linear relationship when re-expressed as $(\ln x, \ln y)$ pairs. We now illustrate that data pairs (x, y) that fit a logarithmic or exponential regression model can also be *linearized* through *logarithmic re-expression*.

Regression Models Related by Logarithmic Re-expression

- **Linear regression:** $y = ax + b$
- **Natural logarithmic regression:** $y = a + b \ln x$
- **Exponential Regression:** $y = a \cdot b^x$
- **Power regression:** $y = a \cdot x^b$

When we examine a scatter plot of data pairs (x, y), we can ask whether one of these four regression models could be the best choice. If the data plot appears to be linear, a linear regression may be the best choice. But when it is visually evident that the data plot is not linear, the best choice may be a natural logarithmic, exponential, or power regression.

Knowing the shapes of logarithmic, exponential, and power function graphs helps us choose an appropriate model. In addition, it is often helpful to re-express the (x, y) data pairs as $(\ln x, y)$, $(x, \ln y)$, or $(\ln x, \ln y)$ and create scatter plots of the re-expressed data. If any of the scatter plots appear to be linear, then we have a likely choice for an appropriate model. See page 299.

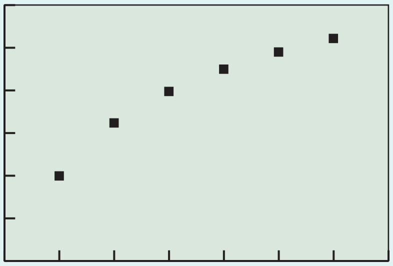
The three regression models can be justified algebraically. We give the justification for exponential regression, and leave the other two justifications as exercises.

$$v = ax + b$$
$$\ln y = ax + b$$
$$y = e^{ax+b} \quad \text{Change to exponential form.}$$
$$y = e^{ax} \cdot e^b \quad \text{Use the laws of exponents.}$$
$$y = e^b \cdot (e^a)^x$$
$$y = c \cdot d^x \quad \text{Let } c = e^b \text{ and } d = e^a.$$

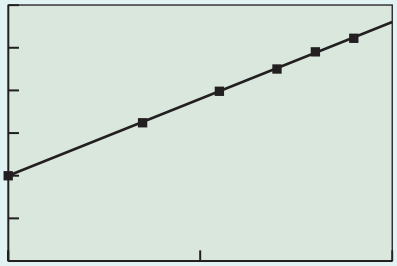
Example 9 illustrates how knowledge about the shapes of logarithmic, exponential, and power function graphs is used in combination with logarithmic re-expression to choose a curve of best fit.

Three Types of Logarithmic Re-expression

1. Natural Logarithmic Regression Re-expressed: $(x, y) \rightarrow (\ln x, y)$



[0, 7] by [0, 30]
(x, y) data
(a)

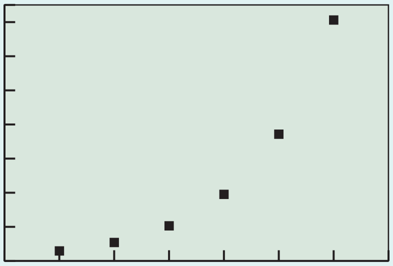


[0, 2] by [0, 30]
$(\ln x, y) = (u, y)$ data with linear regression model $y = au + b$
(b)

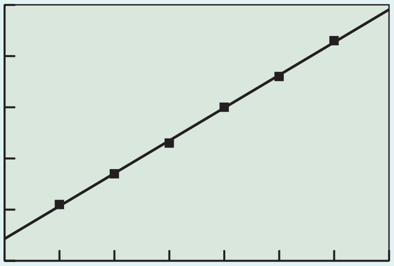
Conclusion:

$y = a \ln x + b$ is the logarithmic regression model for the (x, y) data.

2. Exponential Regression Re-expressed: $(x, y) \rightarrow (x, \ln y)$



[0, 7] by [0, 75]
(x, y) data
(a)

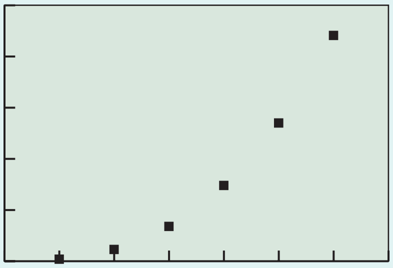


[0, 7] by [0, 5]
$(x, \ln y) = (x, v)$ data with linear regression model $v = ax + b$
(b)

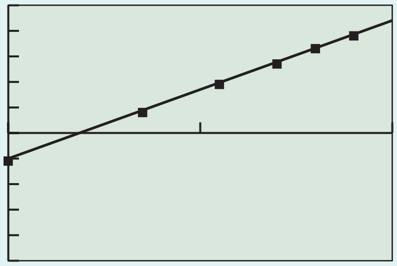
Conclusion:

$y = c(d^x)$, where $c = e^b$ and $d = e^a$, is the exponential regression model for the (x, y) data.

3. Power Regression Re-expressed: $(x, y) \rightarrow (\ln x, \ln y)$



[0, 7] by [0, 50]
(x, y) data
(a)



[0, 2] by [–5, 5]
$(\ln x, \ln y) = (u, v)$ data with linear regression model $v = au + b$
(b)

Conclusion:

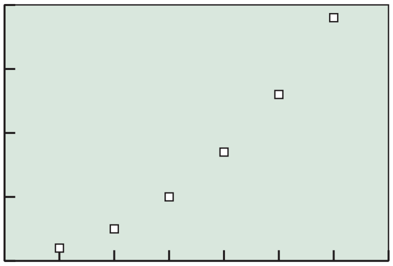
$y = c(x^a)$, where $c = e^b$, is the power regression model for the (x, y) data.

EXAMPLE 9 Selecting a Regression Model

Decide whether these data can be best modeled by logarithmic, exponential, or power regression. Find the appropriate regression model.

x	1	2	3	4	5	6
y	2	5	10	17	26	38

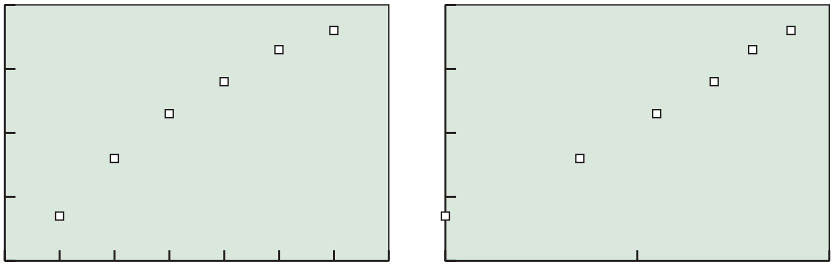
SOLUTION The shape of the data plot in Figure 3.38 suggests that the data could be modeled by an exponential or power function, but not a logarithmic function.



[0, 7] by [0, 40]

FIGURE 3.38 A scatter plot of the original data of Example 9.

Figure 3.39a shows the $(x, \ln y)$ plot, and Figure 3.39b shows the $(\ln x, \ln y)$ plot. Of these two plots, the $(\ln x, \ln y)$ plot appears to be more linear, so we find the power regression model for the original data.

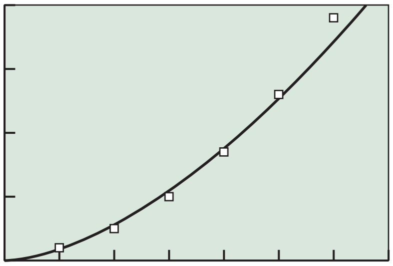


[0, 7] by [0, 4]
$(x, \ln y)$
(a)

[0, 2] by [0, 4]
$(\ln x, \ln y)$
(b)

FIGURE 3.39 Two logarithmic re-expressions of the data of Example 9.

Figure 3.40 shows the scatter plot of the original (x, y) data with the graph of the power regression model $y = 1.7910x^{1.6472}$ superimposed.



[0, 7] by [0, 40]

FIGURE 3.40 A power regression model fits the data of Example 9.

Now try Exercise 55.

QUICK REVIEW 3.5 *(For help, go to Sections P.1 and 1.4.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, prove that each function in the given pair is the inverse of the other.

1. $f(x) = e^{2x}$ and $g(x) = \ln(x^{1/2})$

2. $f(x) = 10^{x/2}$ and $g(x) = \log x^2, x > 0$

3. $f(x) = (1/3)\ln x$ and $g(x) = e^{3x}$

4. $f(x) = 3\log x^2, x > 0$ and $g(x) = 10^{x/6}$

In Exercises 5 and 6, write the number in scientific notation.

5. The mean distance from Jupiter to the Sun is about 778,300,000 km.

6. An atomic nucleus has a diameter of about 0.000000000000001 m.

In Exercises 7 and 8, write the number in decimal form.

7. Avogadro's number is about 6.02×10^{23}.

8. The atomic mass unit is about 1.66×10^{-27} kg.

In Exercises 9 and 10, use scientific notation to simplify the expression; leave your answer in scientific notation.

9. $(186{,}000)(31{,}000{,}000)$

10. $\dfrac{0.0000008}{0.000005}$

SECTION 3.5 EXERCISES

In Exercises 1–10, find the exact solution algebraically, and check it by substituting into the original equation.

1. $36\left(\frac{1}{3}\right)^{x/5} = 4$

2. $32\left(\frac{1}{4}\right)^{x/3} = 2$

3. $2 \cdot 5^{x/4} = 250$

4. $3 \cdot 4^{x/2} = 96$

5. $2(10^{-x/3}) = 20$

6. $3(5^{-x/4}) = 15$

7. $\log x = 4$

8. $\log_2 x = 5$

9. $\log_4 (x - 5) = -1$

10. $\log_4 (1 - x) = 1$

In Exercises 11–18, solve each equation algebraically. Obtain a numerical approximation for your solution and check it by substituting into the original equation.

11. $1.06^x = 4.1$

12. $0.98^x = 1.6$

13. $50e^{0.035x} = 200$

14. $80e^{0.045x} = 240$

15. $3 + 2e^{-x} = 6$

16. $7 - 3e^{-x} = 2$

17. $3 \ln (x - 3) + 4 = 5$

18. $3 - \log (x + 2) = 5$

In Exercises 19–24, state the domain of each function. Then match the function with its graph. (Each graph shown has a window of $[-4.7, 4.7]$ by $[-3.1, 3.1]$).

19. $f(x) = \log [x(x + 1)]$

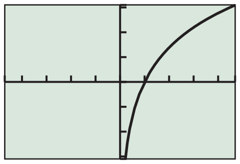
20. $g(x) = \log x + \log (x + 1)$

21. $f(x) = \ln \frac{x}{x + 1}$

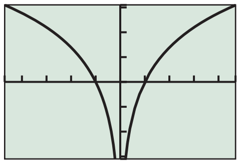
22. $g(x) = \ln x - \ln (x + 1)$

23. $f(x) = 2 \ln x$

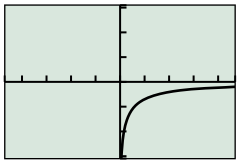
24. $g(x) = \ln x^2$



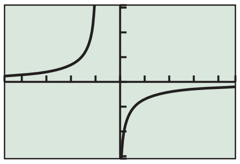
(a)



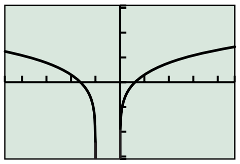
(b)



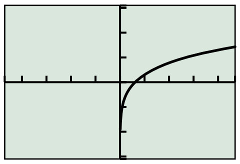
(c)



(d)



(e)



(f)

In Exercises 25–38, solve each equation by the method of your choice. Support your solution by a second method.

25. $\log x^2 = 6$

26. $\ln x^2 = 4$

27. $\log x^4 = 2$

28. $\ln x^6 = 12$

29. $\frac{2^x - 2^{-x}}{3} = 4$

30. $\frac{2^x + 2^{-x}}{2} = 3$

31. $\frac{e^x + e^{-x}}{2} = 4$

32. $2e^{2x} + 5e^x - 3 = 0$

33. $\frac{500}{1 + 25e^{0.3x}} = 200$

34. $\frac{400}{1 + 95e^{-0.6x}} = 150$

35. $\frac{1}{2} \ln (x + 3) - \ln x = 0$

36. $\log x - \frac{1}{2} \log (x + 4) = 1$

37. $\ln (x - 3) + \ln (x + 4) = 3 \ln 2$

38. $\log (x - 2) + \log (x + 5) = 2 \log 3$

In Exercises 39–44, determine by how many orders of magnitude the quantities differ.

39. A \$100 bill and a dime

40. A canary weighing 20 g and a hen weighing 2 kg

41. An earthquake rated 7 on the Richter scale and one rated 5.5

42. Lemon juice with pH $= 2.3$ and beer with pH $= 4.1$

43. The sound intensities of a riveter at 95 dB and ordinary conversation at 65 dB

44. The sound intensities of city traffic at 70 dB and rustling leaves at 10 dB

45. Comparing Earthquakes How many times more severe was the 1978 Mexico City earthquake ($R = 7.9$) than the 1994 Los Angeles earthquake ($R = 6.6$)?

46. Comparing Earthquakes How many times more severe was the 1995 Kobe, Japan, earthquake ($R = 7.2$) than the 1994 Los Angeles earthquake ($R = 6.6$)?

47. Chemical Acidity The pH of carbonated water is 3.9 and the pH of household ammonia is 11.9.

(a) What are their hydrogen-ion concentrations?

(b) How many times greater is the hydrogen-ion concentration of carbonated water than that of ammonia?

(c) By how many orders of magnitude do the concentrations differ?

48. Chemical Acidity Stomach acid has a pH of about 2.0, and blood has a pH of 7.4.

(a) What are their hydrogen-ion concentrations?

(b) How many times greater is the hydrogen-ion concentration of stomach acid than that of blood?

(c) By how many orders of magnitude do the concentrations differ?

49. Newton's Law of Cooling A cup of coffee has cooled from 92°C to 50°C after 12 min in a room at 22°C. How long will the cup take to cool to 30°C?

50. Newton's Law of Cooling A cake is removed from an oven at 350°F and cools to 120°F after 20 min in a room at 65°F. How long will the cake take to cool to 90°F?

51. Newton's Law of Cooling Experiment A thermometer is removed from a cup of coffee and placed in water with a temperature (T_m) of 10°C. The data in Table 3.24 were collected over the next 30 sec.

Table 3.24 Experimental Data

Time t	Temp T	$T - T_m$
2	80.47	70.47
5	69.39	59.39
10	49.66	39.66
15	35.26	25.26
20	28.15	18.15
25	23.56	13.56
30	20.62	10.62

(a) Draw a scatter plot of the data $T - T_m$.

(b) Find an exponential regression equation for the $T - T_m$ data. Superimpose its graph on the scatter plot.

(c) Estimate the thermometer reading when it was removed from the coffee.

52. Newton's Law of Cooling Experiment A thermometer was removed from a cup of hot chocolate and placed in a saline solution with temperature $T_m = 0$°C. The data in Table 3.25 were collected over the next 30 sec.

(a) Draw a scatter plot of the data $T - T_m$.

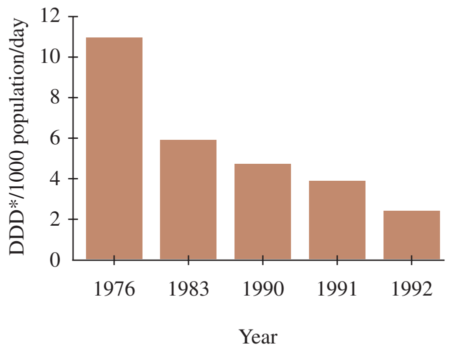
(b) Find an exponential regression equation for the $T - T_m$ data. Superimpose its graph on the scatter plot.

(c) Estimate the thermometer reading when it was removed from the hot chocolate.

Table 3.25 Experimental Data

Time t	Temp T	$T - T_m$
2	74.68	74.68
5	61.99	61.99
10	34.89	34.89
15	21.95	21.95
20	15.36	15.36
25	11.89	11.89
30	10.02	10.02

53. Penicillin Use The use of penicillin became so widespread in the 1980s in Hungary that it became practically useless against common sinus and ear infections. Now the use of more effective antibiotics has caused a decline in penicillin resistance. The bar graph shows the use of penicillin in Hungary for selected years.



*Defined Daily Dose

Source: Science, vol. 264, April 15, 1994, American Association for the Advancement of Science.

(a) From the bar graph we read the data pairs to be approximately $(1, 11)$, $(8, 6)$, $(15, 4.8)$, $(16, 4)$, and $(17, 2.5)$, using $t = 1$ for 1976, $t = 8$ for 1983, and so on. Complete a scatter plot for these data.

(b) Writing to Learn Discuss whether the bar graph shown or the scatter plot that you completed best represents the data and why.

54. Writing to Learn Which regression model would you use for the data in Exercise 53? Discuss various options, and explain why you chose the model you did. Support your writing with tables and graphs as needed.

Writing to Learn In Exercises 55–58, tables of (x, y) data pairs are given. Determine whether a linear, logarithmic, exponential, or power regression equation is the best model for the data. Explain your choice. Support your writing with tables and graphs as needed.

55.

x	1	2	3	4
y	3	4.4	5.2	5.8

56.

x	1	2	3	4
y	6	18	54	162

57.

x	1	2	3	4
y	3	6	12	24

58.

x	1	2	3	4
y	5	7	9	11

Standardized Test Questions

59. True or False The order of magnitude of a positive number is its natural logarithm. Justify your answer.

60. True or False According to Newton's Law of Cooling, an object will approach the temperature of the medium that surrounds it. Justify your answer.

In Exercises 61–64, solve the problem without using a calculator.

61. Multiple Choice Solve $2^{3x-1} = 32$.

(A) $x = 1$ **(B)** $x = 2$ **(C)** $x = 4$

(D) $x = 11$ **(E)** $x = 13$

62. Multiple Choice Solve $\ln x = -1$.

(A) $x = -1$ **(B)** $x = 1/e$ **(C)** $x = 1$

(D) $x = e$ **(E)** No solution is possible.

63. Multiple Choice How many times more severe was the 2001 earthquake in Arequipa, Peru $(R_1 = 8.1)$, than the 1998 double earthquake in Takhar province, Afghanistan $(R_2 = 6.1)$?

(A) 2 **(B)** 6.1 **(C)** 8.1

(D) 14.2 **(E)** 100

64. Multiple Choice Newton's Law of Cooling is

(A) an exponential model. **(B)** a linear model.

(C) a logarithmic model. **(D)** a logistic model.

(E) a power model.

Explorations

In Exercises 65 and 66, use the data in Table 3.26. Determine whether a linear, logarithmic, exponential, power, or logistic regression equation is the best model for the data. Explain your choice. Support your writing with tables and graphs as needed.

Table 3.26 Populations of Two U.S. States (in thousands)

Year	Alaska	Hawaii
1900	63.6	154
1910	64.4	192
1920	55.0	256
1930	59.2	368
1940	72.5	423
1950	128.6	500
1960	226.2	633
1970	302.6	770
1980	401.9	965
1990	550.0	1108
2000	626.9	1212

Source: U.S. Census Bureau.

65. Writing to Learn Modeling Population Which regression equation is the best model for Alaska's population?

66. Writing to Learn Modeling Population Which regression equation is the best model for Hawaii's population?

67. Group Activity Normal Distribution The function

$$f(x) = k \cdot e^{-cx^2},$$

where c and k are positive constants, is a bell-shaped curve that is useful in probability and statistics.

(a) Graph f for $c = 1$ and $k = 0.1, 0.5, 1, 2, 10$. Explain the effect of changing k.

(b) Graph f for $k = 1$ and $c = 0.1, 0.5, 1, 2, 10$. Explain the effect of changing c.

Extending the Ideas

68. Writing to Learn Prove if $u/v = 10^n$ for $u > 0$ and $v > 0$, then $\log u - \log v = n$. Explain how this result relates to powers of ten and orders of magnitude.

69. Potential Energy The potential energy E (the energy stored for use at a later time) between two ions in a certain molecular structure is modeled by the function

$$E = -\frac{5.6}{r} + 10e^{-r/3}$$

where r is the distance separating the nuclei.

(a) Writing to Learn Graph this function in the window $[-10, 10]$ by $[-10, 30]$, and explain which portion of the graph does not represent this potential energy situation.

(b) Identify a viewing window that shows that portion of the graph (with $r \le 10$) which represents this situation, and find the maximum value for E.

70. In Example 8, the Newton's Law of Cooling model was

$$T(t) - T_m = (T_0 - T_m)e^{-kt} = 61.656 \times 0.92770^t.$$

Determine the value of k.

71. Justify the conclusion made about natural logarithmic regression on page 299.

72. Justify the conclusion made about power regression on page 299.

In Exercises 73–78, solve the equation or inequality.

73. $e^x + x = 5$

74. $e^{2x} - 8x + 1 = 0$

75. $e^x < 5 + \ln x$

76. $\ln |x| - e^{2x} \ge 3$

77. $2 \log x - 4 \log 3 > 0$

78. $2 \log (x + 1) - 2 \log 6 < 0$