3.3 Logarithmic Functions and Their Graphs

What you'll learn about

- Inverses of Exponential Functions
- Common Logarithms—Base 10
- Natural Logarithms—Base *e*
- Graphs of Logarithmic Functions
- Measuring Sound Using Decibels

... and why

Logarithmic functions are used in many applications, including the measurement of the relative intensity of sounds.

Inverses of Exponential Functions

In Section 1.4 we learned that, if a function passes the *horizontal line test*, then the inverse of the function is also a function. Figure 3.18 shows that an exponential function $f(x) = b^x$ would pass the horizontal line test. So it has an inverse that is a function. This inverse is the **logarithmic function with base *b***, denoted $\log_b(x)$, or more simply as $\log_b x$. That is, if $f(x) = b^x$ with $b > 0$ and $b \neq 1$, then $f^{-1}(x) = \log_b x$. See Figure 3.19.

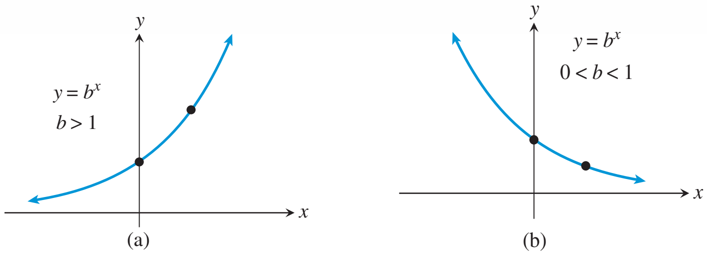


FIGURE 3.18 Exponential functions are either (a) increasing or (b) decreasing.

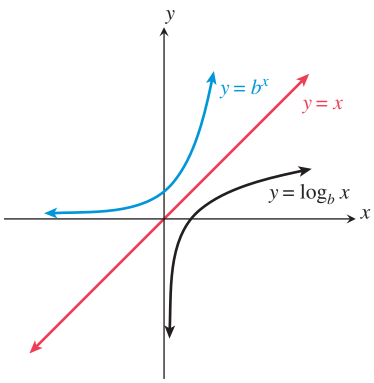


FIGURE 3.19 Because logarithmic functions are inverses of exponential functions, we can obtain the graph of a logarithmic function by the mirror or rotational methods discussed in Section 1.4.

An immediate and useful consequence of this definition is the link between an exponential equation and its logarithmic counterpart.

Changing Between Logarithmic and Exponential Form

If $x > 0$ and $0 < b \neq 1$, then

$$y = \log_b(x) \quad \text{if and only if} \quad b^y = x.$$

This linking statement says that *a logarithm is an exponent*. Because logarithms are exponents, we can evaluate simple logarithmic expressions using our understanding of exponents.

A Bit of History

Logarithmic functions were developed around 1594 as computational tools by Scottish mathematician John Napier (1550–1617). He originally called them "artificial numbers," but changed the name to logarithms, which means "reckoning numbers."

EXAMPLE 1 Evaluating Logarithms

(a) $\log_2 8 = 3$ because $2^3 = 8$.

(b) $\log_3 \sqrt{3} = 1/2$ because $3^{1/2} = \sqrt{3}$.

(c) $\log_5 \dfrac{1}{25} = -2$ because $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$.

(d) $\log_4 1 = 0$ because $4^0 = 1$.

(e) $\log_7 7 = 1$ because $7^1 = 7$.

Now try Exercise 1.

We can generalize the relationships observed in Example 1.

Generally $b > 1$

In practice, logarithmic bases are almost always greater than 1.

Basic Properties of Logarithms

For $0 < b \neq 1$, $x > 0$, and any real number y,

- $\log_b 1 = 0$ because $b^0 = 1$.
- $\log_b b = 1$ because $b^1 = b$.
- $\log_b b^y = y$ because $b^y = b^y$.
- $b^{\log_b x} = x$ because $\log_b x = \log_b x$.

These properties give us efficient ways to evaluate simple logarithms and some exponential expressions. The first two parts of Example 2 are the same as the first two parts of Example 1.

EXAMPLE 2 Evaluating Logarithmic and Exponential Expressions

(a) $\log_2 8 = \log_2 2^3 = 3.$

(b) $\log_3 \sqrt{3} = \log_3 3^{1/2} = 1/2.$

(c) $6^{\log_6 11} = 11.$ *Now try Exercise 5.*

Logarithmic functions are inverses of exponential functions. So the inputs and outputs are switched. Table 3.16 illustrates this relationship for $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$.

Table 3.16 An Exponential Function and Its Inverse

x	$f(x) = 2^x$	x	$f^{-1}(x) = \log_2 x$
-3	1/8	1/8	-3
-2	1/4	1/4	-2
-1	1/2	1/2	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

This relationship can be used to produce both tables and graphs for logarithmic functions, as you will discover in Exploration 1.

EXPLORATION 1 Comparing Exponential and Logarithmic Functions

1. Set your grapher to Parametric mode and Simultaneous graphing mode.

   Set X1T = T and Y1T = 2^T.

   Set X2T = 2^T and Y2T = T.

   Creating Tables. Set TblStart = -3 and ΔTbl = 1. Use the Table feature of your grapher to obtain the decimal form of both parts of Table 3.16. Be sure to scroll to the right to see X2T and Y2T.

   Drawing Graphs. Set Tmin = -6, Tmax = 6, and Tstep = 0.5. Set the (x, y) window to $[-6, 6]$ by $[-4, 4]$. Use the Graph feature to obtain the simultaneous graphs of $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$. Use the Trace feature to explore the numerical relationships within the graphs.

2. *Graphing in Function mode.* Graph $y = 2^x$ in the same window. Then use the "draw inverse" command to draw the graph of $y = \log_2 x$.

Common Logarithms—Base 10

Logarithms with base 10 are called **common logarithms**. Because of their connection to our base-ten number system, the metric system, and scientific notation, common logarithms are especially useful. We often drop the subscript of 10 for the base when using common logarithms. The common logarithmic function $\log_{10} x = \log x$ is the inverse of the exponential function $f(x) = 10^x$. So

$$y = \log x \quad \text{if and only if} \quad 10^y = x.$$

Applying this relationship, we can obtain other relationships for logarithms with base 10.

Basic Properties of Common Logarithms

Let x and y be real numbers with $x > 0$.

- $\log 1 = 0$ because $10^0 = 1$.
- $\log 10 = 1$ because $10^1 = 10$.
- $\log 10^y = y$ because $10^y = 10^y$.
- $10^{\log x} = x$ because $\log x = \log x$.

Some Words of Warning

In Figure 3.20, notice we used "10^Ans" instead of "10^1.537819095" to check $\log(34.5)$. This is because graphers generally store more digits than they display and so we can obtain a more accurate check. Even so, because $\log(34.5)$ is an irrational number, a grapher cannot produce its exact value, so checks like those shown in Figure 3.20 may not always work out so perfectly.

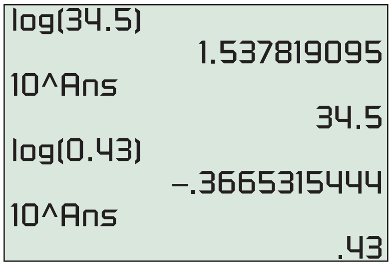


FIGURE 3.20 Doing and checking common logarithmic computations. (Example 4)

Using the definition of common logarithm or these basic properties, we can evaluate expressions involving a base of 10.

EXAMPLE 3 Evaluating Logarithmic and Exponential Expressions—Base 10

(a) $\log 100 = \log_{10} 100 = 2$ because $10^2 = 100$.

(b) $\log \sqrt[5]{10} = \log 10^{1/5} = \dfrac{1}{5}$.

(c) $\log \dfrac{1}{1000} = \log \dfrac{1}{10^3} = \log 10^{-3} = -3$.

(d) $10^{\log 6} = 6$. ***Now try Exercise 7.***

Common logarithms can be evaluated by using the $\boxed{\text{LOG}}$ key on a calculator, as illustrated in Example 4.

EXAMPLE 4 Evaluating Common Logarithms with a Calculator

Use a calculator to evaluate the logarithmic expression if it is defined, and check your result by evaluating the corresponding exponential expression.

(a) $\log 34.5 = 1.537\ldots$ because $10^{1.537\ldots} = 34.5$.

(b) $\log 0.43 = -0.366\ldots$ because $10^{-0.366\ldots} = 0.43$.

See Figure 3.20.

(c) $\log(-3)$ is undefined because there is no real number y such that $10^y = -3$. A grapher will yield either an error message or a complex-number answer for entries such as $\log(-3)$. We shall restrict the domain of logarithmic functions to the set of positive real numbers and ignore such complex-number answers.

Now try Exercise 25.

Changing from logarithmic form to exponential form sometimes is enough to solve an equation involving logarithmic functions.

EXAMPLE 5 Solving Simple Logarithmic Equations

Solve each equation by changing it to exponential form.

(a) $\log x = 3$ **(b)** $\log_2 x = 5$

SOLUTION

(a) Changing to exponential form, $x = 10^3 = 1000$.

(b) Changing to exponential form, $x = 2^5 = 32$. ***Now try Exercise 33.***

Reading a Natural Log

The expression $\ln x$ is pronounced "el en of ex." The "l" is for logarithm, and the "n" is for natural.

Natural Logarithms—Base e

Because of their special calculus properties, logarithms with the natural base e are used in many situations. Logarithms with base e are **natural logarithms**. We often use the special abbreviation "ln" (without a subscript) to denote a natural logarithm. Thus, the natural logarithmic function $\log_e x = \ln x$. It is the inverse of the exponential function $f(x) = e^x$. So

$$y = \ln x \quad \text{if and only if} \quad e^y = x.$$

Applying this relationship, we can obtain other fundamental relationships for logarithms with the natural base e.

Basic Properties of Natural Logarithms

Let x and y be real numbers with $x > 0$.

- $\ln 1 = 0$ because $e^0 = 1$.
- $\ln e = 1$ because $e^1 = e$.
- $\ln e^y = y$ because $e^y = e^y$.
- $e^{\ln x} = x$ because $\ln x = \ln x$.

Using the definition of natural logarithm or these basic properties, we can evaluate expressions involving the natural base e.

EXAMPLE 6 Evaluating Logarithmic and Exponential Expressions—Base e

(a) $\ln \sqrt{e} = \log_e \sqrt{e} = 1/2$ because $e^{1/2} = \sqrt{e}$.

(b) $\ln e^5 = \log_e e^5 = 5$.

(c) $e^{\ln 4} = 4$. *Now try Exercise 13.*

Natural logarithms can be evaluated by using the $\boxed{\text{LN}}$ key on a calculator, as illustrated in Example 7.

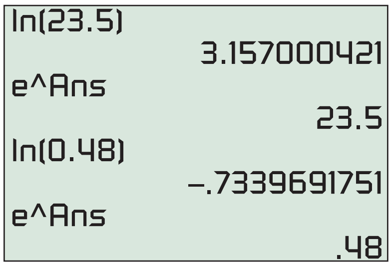


FIGURE 3.21 Doing and checking natural logarithmic computations. (Example 7)

EXAMPLE 7 Evaluating Natural Logarithms with a Calculator

Use a calculator to evaluate the logarithmic expression, if it is defined, and check your result by evaluating the corresponding exponential expression.

(a) $\ln 23.5 = 3.157\ldots$ because $e^{3.157\ldots} = 23.5$.

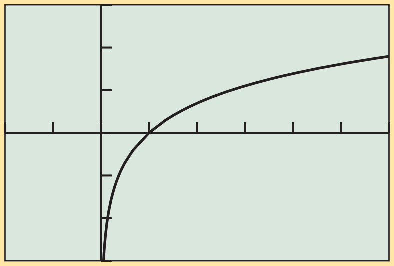
(b) $\ln 0.48 = -0.733\ldots$ because $e^{-0.733\ldots} = 0.48$.

See Figure 3.21.

(c) $\ln(-5)$ is undefined because there is no real number y such that $e^y = -5$. A grapher will yield either an error message or a complex-number answer for entries such as $\ln(-5)$. We will continue to restrict the domain of logarithmic functions to the set of positive real numbers and ignore such complex-number answers. *Now try Exercise 29.*

Graphs of Logarithmic Functions

The natural logarithmic function $f(x) = \ln x$ is one of the basic functions introduced in Section 1.3. We now list its properties.



[–2, 6] by [–3, 3]

FIGURE 3.22

BASIC FUNCTION The Natural Logarithmic Function

$f(x) = \ln x$
Domain: $(0, \infty)$
Range: All reals
Continuous on $(0, \infty)$
Increasing on $(0, \infty)$
No symmetry
Not bounded above or below
No local extrema
No horizontal asymptotes
Vertical asymptote: $x = 0$
End behavior: $\lim_{x \to \infty} \ln x = \infty$

Any logarithmic function $g(x) = \log_b x$ with $b > 1$ has the same domain, range, continuity, increasing behavior, lack of symmetry, and other general behavior as $f(x) = \ln x$. It is rare that we are interested in logarithmic functions $g(x) = \log_b x$ with $0 < b < 1$. So, the graph and behavior of $f(x) = \ln x$ are typical of logarithmic functions.

We now consider the graphs of the common and natural logarithmic functions and their geometric transformations. To understand the graphs of $y = \log x$ and $y = \ln x$, we can compare each to the graph of its inverse, $y = 10^x$ and $y = e^x$, respectively. Figure 3.23a shows that the graphs of $y = \ln x$ and $y = e^x$ are reflections of each other across the line $y = x$. Similarly, Figure 3.23b shows that the graphs of $y = \log x$ and $y = 10^x$ are reflections of each other across this same line.

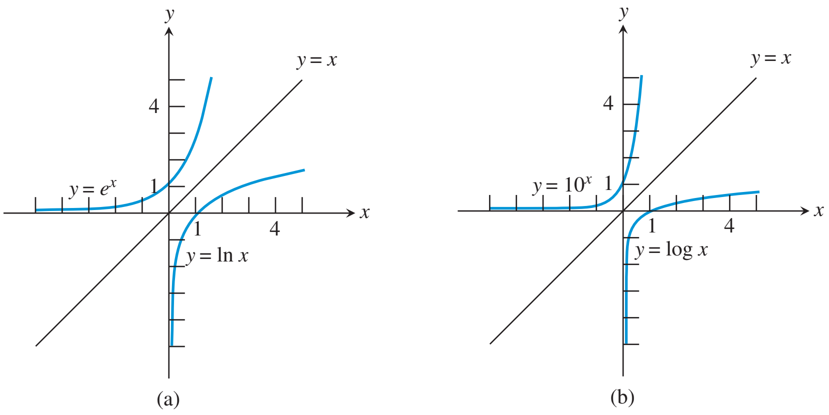
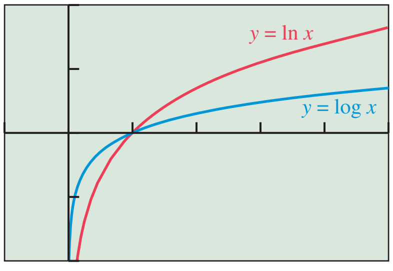


FIGURE 3.23 Two pairs of inverse functions.



[–1, 5] by [–2, 2]

FIGURE 3.24 The graphs of the common and natural logarithmic functions.

From Figure 3.24 we can see that the graphs of $y = \log x$ and $y = \ln x$ have much in common. Figure 3.24 also shows how they differ.

The geometric transformations studied in Section 1.5, together with our knowledge of the graphs of $y = \ln x$ and $y = \log x$, allow us to predict the graphs of the functions in Example 8.

EXAMPLE 8 Transforming Logarithmic Graphs

Describe how to transform the graph of $y = \ln x$ or $y = \log x$ into the graph of the given function.

(a) $g(x) = \ln (x + 2)$ **(b)** $h(x) = \ln (3 - x)$

(c) $g(x) = 3 \log x$ **(d)** $h(x) = 1 + \log x$

SOLUTION

(a) The graph of $g(x) = \ln (x + 2)$ is obtained by translating the graph of $y = \ln (x)$ 2 units to the left. See Figure 3.25a.

(b) $h(x) = \ln (3 - x) = \ln [-(x - 3)]$. So we obtain the graph of $h(x) = \ln (3 - x)$ from the graph of $y = \ln x$ by applying, in order, a reflection across the y-axis followed by a translation 3 units to the right. See Figure 3.25b.

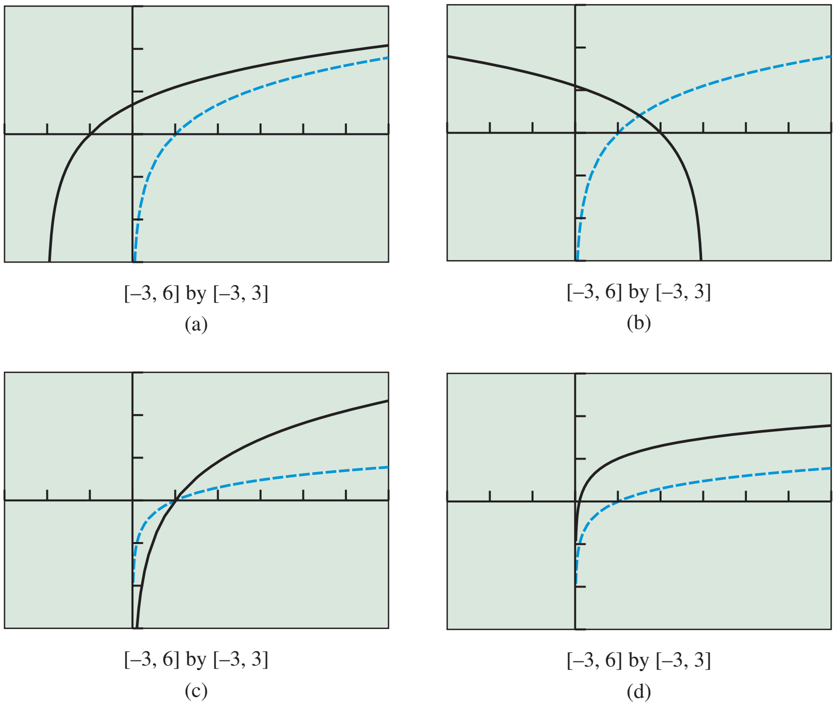


FIGURE 3.25 Transforming $y = \ln x$ to obtain (a) $g(x) = \ln (x + 2)$ and (b) $h(x) = \ln (3 - x)$; and $y = \log x$ to obtain (c) $g(x) = 3 \log x$ and (d) $h(x) = 1 + \log x$. (Example 8)

(c) The graph of $g(x) = 3 \log x$ is obtained by vertically stretching the graph of $f(x) = \log x$ by a factor of 3. See Figure 3.25c.

(d) We can obtain the graph of $h(x) = 1 + \log x$ from the graph of $f(x) = \log x$ by a translation 1 unit up. See Figure 3.25d. ***Now try Exercise 41.***

Sound Intensity

Sound intensity is the energy per unit time of a sound wave over a given area, and is measured in watts per square meter (W/m^2).

Measuring Sound Using Decibels

Table 3.17 lists assorted sounds. Notice that a jet at takeoff is 100 trillion times as loud as a soft whisper. Because the range of audible sound intensities is so great, common logarithms (powers of 10) are used to compare how loud sounds are.

Table 3.17 Approximate Intensities of Selected Sounds

Sound	Intensity (W/m^2)
Hearing threshold	10^{-12}
Soft whisper at 5 m	10^{-11}
City traffic	10^{-5}
Subway train	10^{-2}
Pain threshold	10^{0}
Jet at takeoff	10^{3}

Source: Adapted from R. W. Reading, Exploring Physics: Concepts and Applications. Belmont, CA: Wadsworth, 1984.

Bel Is for Bell

The original unit for sound intensity level was the *bel* (B), which proved to be inconveniently large. So the decibel, one-tenth of a bel, has replaced it. The bel was named in honor of Scottish-born American Alexander Graham Bell (1847–1922), inventor of the telephone.

DEFINITION Decibels

The level of sound intensity in **decibels** (dB) is

$$\beta = 10 \log(I/I_0),$$

where β (beta) is the number of decibels, I is the sound intensity in W/m^2, and $I_0 = 10^{-12}$ W/m^2 is the threshold of human hearing (the quietest audible sound intensity).

Chapter Opener Problem (from page 251)

Problem: How loud is a train inside a subway tunnel?

Solution: Based on the data in Table 3.17,

$$\begin{aligned}\beta &= 10 \log(I/I_0)\\ &= 10 \log(10^{-2}/10^{-12})\\ &= 10 \log(10^{10})\\ &= 10 \cdot 10 = 100\end{aligned}$$

So the sound intensity level inside the subway tunnel is 100 dB.

QUICK REVIEW 3.3 *(For help, go to Sections P.1 and A.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–6, evaluate the expression without using a calculator.

1. 5^{-2} **2.** 10^{-3}

3. $\frac{4^0}{5}$ **4.** $\frac{1^0}{2}$

5. $\frac{8^{11}}{2^{28}}$ **6.** $\frac{9^{13}}{27^8}$

In Exercises 7–10, rewrite as a base raised to a rational number exponent.

7. $\sqrt{5}$ **8.** $\sqrt[3]{10}$

9. $\frac{1}{\sqrt{e}}$ **10.** $\frac{1}{\sqrt[3]{e^2}}$

SECTION 3.3 EXERCISES

In Exercises 1–18, evaluate the logarithmic expression without using a calculator.

1. $\log_4 4$ **2.** $\log_6 1$

3. $\log_2 32$ **4.** $\log_3 81$

5. $\log_5 \sqrt[3]{25}$ **6.** $\log_6 \dfrac{1}{\sqrt[5]{36}}$

7. $\log 10^3$ **8.** $\log 10{,}000$

9. $\log 100{,}000$ **10.** $\log 10^{-4}$

11. $\log \sqrt[3]{10}$ **12.** $\log \dfrac{1}{\sqrt{1000}}$

13. $\ln e^3$ **14.** $\ln e^{-4}$

15. $\ln \dfrac{1}{e}$ **16.** $\ln 1$

17. $\ln \sqrt[4]{e}$ **18.** $\ln \dfrac{1}{\sqrt{e^7}}$

In Exercises 19–24, evaluate the expression without using a calculator.

19. $7^{\log_7 3}$ **20.** $5^{\log_5 8}$

21. $10^{\log (0.5)}$ **22.** $10^{\log 14}$

23. $e^{\ln 6}$ **24.** $e^{\ln (1/5)}$

In Exercises 25–32, use a calculator to evaluate the logarithmic expression if it is defined, and check your result by evaluating the corresponding exponential expression.

25. $\log 9.43$ **26.** $\log 0.908$

27. $\log (-14)$ **28.** $\log (-5.14)$

29. $\ln 4.05$ **30.** $\ln 0.733$

31. $\ln (-0.49)$ **32.** $\ln (-3.3)$

In Exercises 33–36, solve the equation by changing it to exponential form.

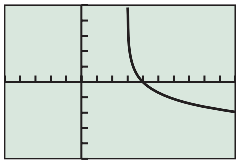
33. $\log x = 2$ **34.** $\log x = 4$

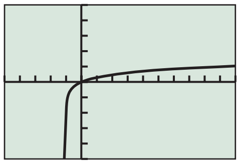
35. $\log x = -1$ **36.** $\log x = -3$

In Exercises 37–40, match the function with its graph.

37. $f(x) = \log (1 - x)$ **38.** $f(x) = \log (x + 1)$

39. $f(x) = -\ln (x - 3)$ **40.** $f(x) = -\ln (4 - x)$


(a)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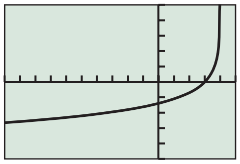
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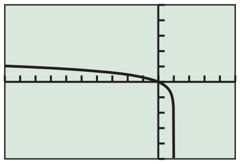
(d)

In Exercises 41–46, describe how to transform the graph of $y = \ln x$ into the graph of the given function. Sketch the graph by hand and support your sketch with a grapher.

41. $f(x) = \ln (x + 3)$ **42.** $f(x) = \ln (x) + 2$

43. $f(x) = \ln (-x) + 3$ **44.** $f(x) = \ln (-x) - 2$

45. $f(x) = \ln (2 - x)$ **46.** $f(x) = \ln (5 - x)$

In Exercises 47–52, describe how to transform the graph of $y = \log x$ into the graph of the given function. Sketch the graph by hand and support with a grapher.

47. $f(x) = -1 + \log (x)$

48. $f(x) = \log (x - 3)$

49. $f(x) = -2 \log (-x)$

50. $f(x) = -3 \log (-x)$

51. $f(x) = 2 \log (3 - x) - 1$

52. $f(x) = -3 \log (1 - x) + 1$

In Exercises 53–58, graph the function, and analyze it for domain, range, continuity, increasing or decreasing behavior, boundedness, extrema, symmetry, asymptotes, and end behavior.

53. $f(x) = \log (x - 2)$ **54.** $f(x) = \ln (x + 1)$

55. $f(x) = -\ln (x - 1)$ **56.** $f(x) = -\log (x + 2)$

57. $f(x) = 3 \log (x) - 1$ **58.** $f(x) = 5 \ln (2 - x) - 3$

59. Sound Intensity Use the data in Table 3.17 to compute the sound intensity in decibels for **(a)** a soft whisper, **(b)** city traffic, and **(c)** a jet at takeoff.

60. Light Absorption The Beer-Lambert Law of Absorption applied to Lake Erie states that the light intensity I (in lumens), at a depth of x feet, satisfies the equation

$$\log \frac{I}{12} = -0.00235x.$$

Find the intensity of the light at a depth of 30 ft.

61. Population Growth Using the data in Table 3.18, compute a logarithmic regression model, and use it to predict when the population of San Antonio will be 1,500,000.

Table 3.18 Population of San Antonio

Year	Population
1970	654,153
1980	785,940
1990	935,933
2000	1,151,305

Source: World Alamanac and Book of Facts 2005.

62. Population Decay Using the data in Table 3.19, compute a logarithmic regression model, and use it to predict when the population of Milwaukee will be 500,000.

Table 3.19 Population of Milwaukee

Year	Population
1970	717,372
1980	636,297
1990	628,088
2000	596,974

Source: World Alamanac and Book of Facts 2005.

Standardized Test Questions

63. True or False A logarithmic function is the inverse of an exponential function. Justify your answer.

64. True or False Common logarithms are logarithms with base 10. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve the problem.

65. Multiple Choice What is the approximate value of the common log of 2?

(A) 0.10523 **(B)** 0.20000

(C) 0.30103 **(D)** 0.69315

(E) 3.32193

66. Multiple Choice Which statement is **false**?

(A) $\log 5 = 2.5 \log 2$ **(B)** $\log 5 = 1 - \log 2$

(C) $\log 5 > \log 2$ **(D)** $\log 5 < \log 10$

(E) $\log 5 = \log 10 - \log 2$

67. Multiple Choice Which statement is **false** about $f(x) = \ln x$?

(A) It is increasing on its domain.

(B) It is symmetric about the origin.

(C) It is continuous on its domain.

(D) It is unbounded.

(E) It has a vertical asymptote.

68. Multiple Choice Which of the following is the inverse of $f(x) = 2 \cdot 3^x$?

(A) $f^{-1}(x) = \log_3 (x/2)$ **(B)** $f^{-1}(x) = \log_2 (x/3)$

(C) $f^{-1}(x) = 2 \log_3 (x)$ **(D)** $f^{-1}(x) = 3 \log_2 (x)$

(E) $f^{-1}(x) = 0.5 \log_3 (x)$

Explorations

69. Writing to Learn Parametric Graphing In the manner of Exploration 1, make tables and graphs for $f(x) = 3^x$ and its inverse $f^{-1}(x) = \log_3 x$. Write a comparative analysis of the two functions regarding domain, range, intercepts, and asymptotes.

70. Writing to Learn Parametric Graphing In the manner of Exploration 1, make tables and graphs for $f(x) = 5^x$ and its inverse $f^{-1}(x) = \log_5 x$. Write a comparative analysis of the two functions regarding domain, range, intercepts, and asymptotes.

71. Group Activity Parametric Graphing In the manner of Exploration 1, find the number $b > 1$ such that the graphs of $f(x) = b^x$ and its inverse $f^{-1}(x) = \log_b x$ have exactly one point of intersection. What is the one point that is in common to the two graphs?

72. Writing to Learn Explain why zero is not in the domain of the logarithmic functions $f(x) = \log_3 x$ and $g(x) = \log_5 x$.

Extending the Ideas

73. Describe how to transform the graph of $f(x) = \ln x$ into the graph of $g(x) = \log_{1/e} x$.

74. Describe how to transform the graph of $f(x) = \log x$ into the graph of $g(x) = \log_{0.1} x$.