3.2 Exponential and Logistic Modeling

What you'll learn about

- Constant Percentage Rate and Exponential Functions
- Exponential Growth and Decay Models
- Using Regression to Model Population
- Other Logistic Models

... and why

Exponential functions model many types of unrestricted growth; logistic functions model restricted growth, including the spread of disease and the spread of rumors.

Constant Percentage Rate and Exponential Functions

Suppose that a population is changing at a **constant percentage rate *r***, where r is the percent rate of change expressed in decimal form. Then the population follows the pattern shown.

Time in Years	Population
0	$P(0) = P_0 =$ initial population
1	$P(1) = P_0 + P_0 r = P_0(1 + r)$
2	$P(2) = P(1) \cdot (1 + r) = P_0(1 + r)^2$
3	$P(3) = P(2) \cdot (1 + r) = P_0(1 + r)^3$
⋮	⋮
t	$P(t) = P_0(1 + r)^t$

So, in this case, the population is an exponential function of time.

Exponential Population Model

If a population P is changing at a constant percentage rate r each year, then

$$P(t) = P_0(1 + r)^t,$$

where P_0 is the initial population, r is expressed as a decimal, and t is time in years.

If $r > 0$, then $P(t)$ is an exponential growth function, and its *growth factor* is the base of the exponential function, $1 + r$.

On the other hand, if $r < 0$, the base $1 + r < 1$, $P(t)$ is an exponential decay function, and $1 + r$ is the *decay factor* for the population.

EXAMPLE 1 Finding Growth and Decay Rates

Tell whether the population model is an exponential growth function or exponential decay function, and find the constant percentage rate of growth or decay.

(a) San Jose: $P(t) = 898{,}759 \cdot 1.0064^t$

(b) Detroit: $P(t) = 1{,}203{,}368 \cdot 0.9858^t$

SOLUTION

(a) Because $1 + r = 1.0064$, $r = 0.0064 > 0$. So, P is an exponential growth function with a growth rate of 0.64%.

(b) Because $1 + r = 0.9858$, $r = -0.0142 < 0$. So, P is an exponential decay function with a decay rate of 1.42%. ***Now try Exercise 1.***

EXAMPLE 2 Finding an Exponential Function

Determine the exponential function with initial value $= 12$, increasing at a rate of 8% per year.

SOLUTION Because $P_0 = 12$ and $r = 8\% = 0.08$, the function is $P(t) = 12(1 + 0.08)^t$ or $P(t) = 12 \cdot 1.08^t$. We could write this as $f(x) = 12 \cdot 1.08^x$, where x represents time. *Now try Exercise 7.*

Exponential Growth and Decay Models

Exponential growth and decay models are used for populations of animals, bacteria, and even radioactive atoms. Exponential growth and decay apply to any situation where the growth is proportional to the current size of the quantity of interest. Such situations are frequently encountered in biology, chemistry, business, and the social sciences.

Exponential growth models can be developed in terms of the time it takes a quantity to double. On the flip side, exponential decay models can be developed in terms of the time it takes for a quantity to be halved. Examples 3 through 5 use these strategies.

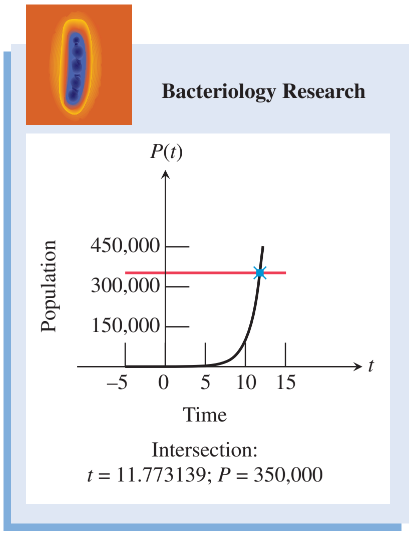


FIGURE 3.12 Rapid growth of a bacteria population. (Example 3)

EXAMPLE 3 Modeling Bacteria Growth

Suppose a culture of 100 bacteria is put into a petri dish and the culture doubles every hour. Predict when the number of bacteria will be 350,000.

SOLUTION

Model

$$
\begin{aligned}
200 &= 100 \cdot 2 && \text{Total bacteria after 1 hr}\\
400 &= 100 \cdot 2^2 && \text{Total bacteria after 2 hr}\\
800 &= 100 \cdot 2^3 && \text{Total bacteria after 3 hr}\\
&\vdots\\
P(t) &= 100 \cdot 2^t && \text{Total bacteria after } t \text{ hr}
\end{aligned}
$$

So the function $P(t) = 100 \cdot 2^t$ represents the bacteria population t hr after it is placed in the petri dish.

Solve Graphically Figure 3.12 shows that the population function intersects $y = 350{,}000$ when $t \approx 11.77$.

Interpret The population of the bacteria in the petri dish will be 350,000 in about 11 hr and 46 min. *Now try Exercise 15.*

Exponential decay functions model the amount of a radioactive substance present in a sample. The number of atoms of a specific element that change from a radioactive state to a nonradioactive state is a fixed fraction per unit time. The process is called **radioactive decay**, and the time it takes for half of a sample to change its state is the **half-life** of the radioactive substance.

EXAMPLE 4 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and there are 5 g (grams) present initially. Find the time when there will be 1 g of the substance remaining.

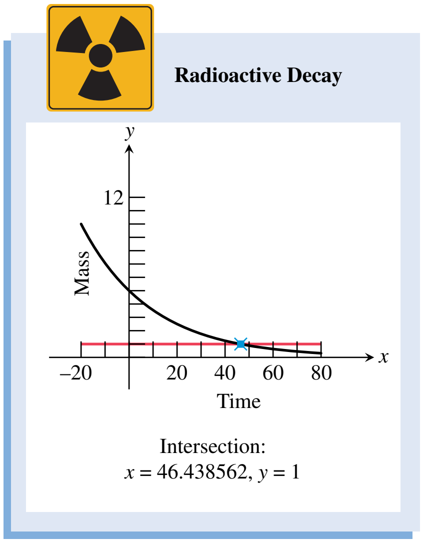


FIGURE 3.13 Radioactive decay. (Example 4)

SOLUTION

Model If t is the time in days, the number of half-lives will be $t/20$.

$$\frac{5}{2} = 5\left(\frac{1}{2}\right)^{20/20} \quad \text{Grams after 20 days}$$

$$\frac{5}{4} = 5\left(\frac{1}{2}\right)^{40/20} \quad \text{Grams after } 2(20) = 40 \text{ days}$$

$$\vdots$$

$$f(t) = 5\left(\frac{1}{2}\right)^{t/20} \quad \text{Grams after } t \text{ days}$$

Thus the function $f(t) = 5 \cdot 0.5^{t/20}$ models the mass in grams of the radioactive substance at time t.

Solve Graphically Figure 3.13 shows that the graph of $f(t) = 5 \cdot 0.5^{t/20}$ intersects $y = 1$ when $t \approx 46.44$.

Interpret There will be 1 g of the radioactive substance left after approximately 46.44 days, or about 46 days, 11 hr. ***Now try Exercise 33.***

Scientists have established that atmospheric pressure at sea level is 14.7 lb/in.^{2}, and the pressure is reduced by half for each 3.6 mi above sea level. For example, the pressure 3.6 mi above sea level is $(1/2)(14.7) = 7.35$ lb/in.^{2}. This rule for atmospheric pressure holds for altitudes up to 50 mi above sea level. Though the context is different, the mathematics of atmospheric pressure closely resembles the mathematics of radioactive decay.

EXAMPLE 5 Determining Altitude from Atmospheric Pressure

Find the altitude above sea level at which the atmospheric pressure is 4 lb/in.^{2}.

SOLUTION

Model

$$7.35 = 14.7 \cdot 0.5^{3.6/3.6} \quad \text{Pressure at 3.6 mi}$$

$$3.675 = 14.7 \cdot 0.5^{7.2/3.6} \quad \text{Pressure at } 2(3.6) = 7.2 \text{ mi}$$

$$\vdots$$

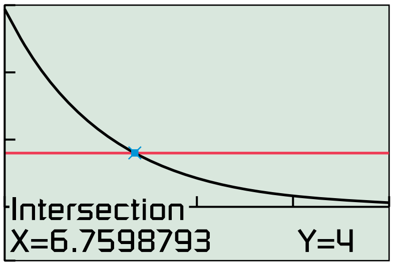
$$P(h) = 14.7 \cdot 0.5^{h/3.6} \quad \text{Pressure at } h \text{ mi}$$

So $P(h) = 14.7 \cdot 0.5^{h/3.6}$ models the atmospheric pressure P (in pounds per square inch) as a function of the height h (in miles above sea level). We must find the value of h that satisfies the equation

$$14.7 \cdot 0.5^{h/3.6} = 4.$$

Solve Graphically Figure 3.14 shows that the graph of $P(h) = 14.7 \cdot 0.5^{h/3.6}$ intersects $y = 4$ when $h \approx 6.76$.

Interpret The atmospheric pressure is 4 lb/in.^{2} at an altitude of approximately 6.76 mi above sea level. ***Now try Exercise 41.***



[0, 20] by [–4, 15]

FIGURE 3.14 A model for atmospheric pressure. (Example 5)

Using Regression to Model Population

So far, our models have been given to us or developed algebraically. We now use exponential and logistic regression to build models from population data.

Due to the post–World War II baby boom and other factors, exponential growth is not a perfect model for the U.S. population. It does, however, provide a means to make approximate predictions, as illustrated in Example 6.

Table 3.9 U.S. Population (in millions)

Year	Population
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.2
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4
2007	301.6

Source: World Almanac and Book of Facts 2009.

EXAMPLE 6 Modeling U.S. Population Using Exponential Regression

Use the 1900–2000 data in Table 3.9 and exponential regression to predict the U.S. population for 2007. Compare the result with the listed value for 2007.

SOLUTION

Model

Let $P(t)$ be the population (in millions) of the United States t years after 1900. Figure 3.15a shows a scatter plot of the data. Using exponential regression, we find a model for the 1990–2000 data:

$$P(t) = 80.5514 \cdot 1.01289^t$$

Figure 3.15b shows the scatter plot of the data with a graph of the population model just found. You can see that the curve fits the data fairly well. The coefficient of determination is $r^2 \approx 0.995$, indicating a close fit and supporting the visual evidence.

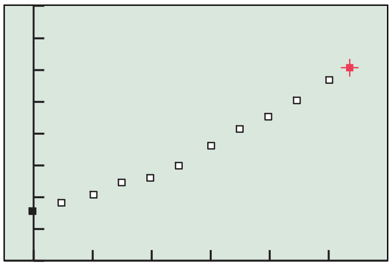
Solve Graphically

To predict the 2007 U.S. population we substitute $t = 107$ into the regression model. Figure 3.15c reports that $P(107) = 80.5514 \cdot 1.01289^{107} \approx 317.1$.

Interpret

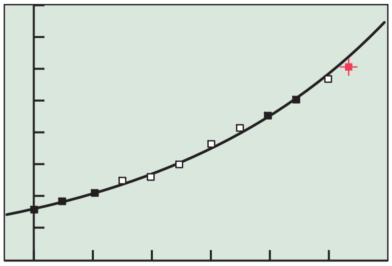
The model predicts the U.S. population was 317.1 million in 2007. The actual population was 301.6 million. We overestimated by 15.5 million, a 5.1% error.

Now try Exercise 43.



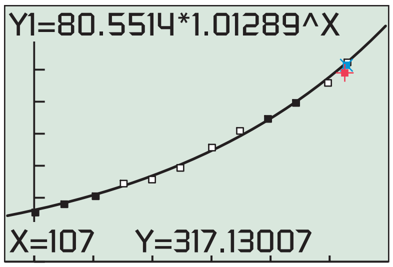
[–10, 120] by [0, 400]

(a)



[–10, 120] by [0, 400]

(b)



[–10, 120] by [0, 400]

(c)

FIGURE 3.15 Scatter plots and graphs for Example 6. The red "+" depicts the data point for 2007. The blue "x" in (c) represents the model's prediction for 2007.

Exponential growth is unrestricted, but population growth often is not. For many populations, the growth begins exponentially, but eventually slows and approaches a limit to growth called the **maximum sustainable population**.

In Section 3.1 we modeled Dallas's population with a logistic function. We now use logistic regression to do the same for the populations of Florida and Pennsylvania. As the data in Table 3.10 suggest, Florida had rapid growth in the second half of the 20th century, whereas Pennsylvania appears to be approaching its maximum sustainable population.

Table 3.10 Populations of Two U.S. States (in millions)

Year	Florida	Pennsylvania
1900	0.5	6.3
1910	0.8	7.7
1920	1.0	8.7
1930	1.5	9.6
1940	1.9	9.9
1950	2.8	10.5
1960	5.0	11.3
1970	6.8	11.8
1980	9.7	11.9
1990	12.9	11.9
2000	16.0	12.3

Source: U.S. Census Bureau.

EXAMPLE 7 Modeling Two States' Populations Using Logistic Regression

Use the data in Table 3.10 and logistic regression to predict the maximum sustainable populations for Florida and Pennsylvania. Graph the logistic models and interpret their significance.

SOLUTION Let $F(t)$ and $P(t)$ be the populations (in millions) of Florida and Pennsylvania, respectively, t years *after 1800*. Figure 3.16a shows a scatter plot of the data for both states; the data for Florida is shown in black, and for Pennsylvania, in red. Using logistic regression, we obtain the models for the two states:

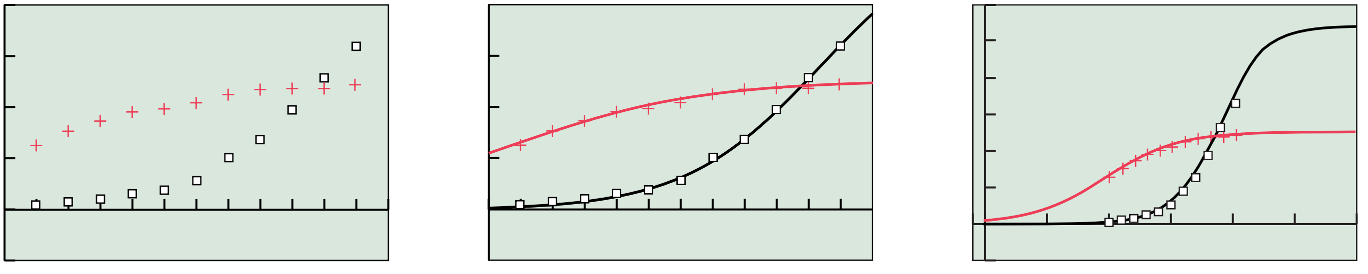
$$F(t) = \frac{28.021}{1 + 9018.63e^{-0.047015t}} \quad \text{and} \quad P(t) = \frac{12.579}{1 + 29.0003e^{-0.034315t}}$$

Figure 3.16b shows the scatter plots of the data with graphs of the two population models. You can see that the curves fit the data fairly well. From the numerators of the models we see that

$$\lim_{t\to\infty} F(t) = 28.021 \quad \text{and} \quad \lim_{t\to\infty} P(t) = 12.579.$$

So the maximum sustainable population for Florida is about 28.0 million, and for Pennsylvania is about 12.6 million.

Figure 3.16c shows a three-century span for the two states. Pennsylvania had rapid growth in the 19th century and first half of the 20th century, and is now approaching its limit to growth. Florida, on the other hand, is currently experiencing extremely rapid growth but should be approaching its maximum sustainable population by the end of the 21st century. ***Now try Exercise 50.***



[90, 210] by [–5, 20]
(a)

[90, 210] by [–5, 20]
(b)

[–10, 300] by [–5, 30]
(c)

FIGURE 3.16 Scatter plots and graphs for Example 7.

Other Logistic Models

In Example 3, the bacteria cannot continue to grow exponentially forever because they cannot grow beyond the confines of the petri dish. In Example 7, though Florida's population is booming now, it will eventually level off, just as Pennsylvania's has done. Sunflowers and many other plants grow to a natural height following a logistic pattern. Chemical acid-base titration curves are logistic. Yeast cultures grow logistically. Contagious diseases and even rumors spread according to logistic models.

EXAMPLE 8 Modeling a Rumor

Watauga High School has 1200 students. Bob, Carol, Ted, and Alice start a rumor, which spreads logistically so that $S(t) = 1200/(1 + 39 \cdot e^{-0.9t})$ models the number of students who have heard the rumor by the end of Day t.

(a) How many students have heard the rumor by the end of Day 0?

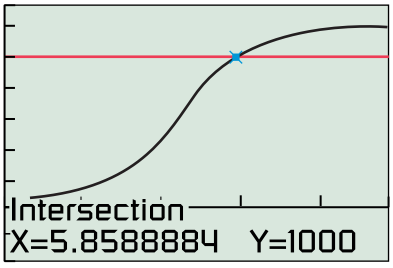
(b) How long does it take for 1000 students to hear the rumor?

SOLUTION

(a) $S(0) = \dfrac{1200}{1 + 39 \cdot e^{-0.9 \cdot 0}} = \dfrac{1200}{1 + 39} = 30.$ So, 30 students have heard the rumor by the end of Day 0.

(b) We need to solve $\dfrac{1200}{1 + 39e^{-0.9t}} = 1000.$

Figure 3.17 shows that the graph of $S(t) = 1200/(1 + 39 \cdot e^{-0.9t})$ intersects $y = 1000$ when $t \approx 5.86$. So toward the end of Day 6 the rumor has reached the ears of 1000 students. *Now try Exercise 45.*



[0, 10] by [–400, 1400]

FIGURE 3.17 The spread of a rumor. (Example 8)

QUICK REVIEW 3.2 *(For help, go to Section P.5.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, convert the percent to decimal form or the decimal into a percent.

1. 15%

2. 0.04

3. Show how to increase 23 by 7% using a single multiplication.

4. Show how to decrease 52 by 4% using a single multiplication.

In Exercises 5 and 6, solve the equation algebraically.

5. $40 \cdot b^2 = 160$

6. $243 \cdot b^3 = 9$

In Exercises 7–10, solve the equation numerically.

7. $782b^6 = 838$

8. $93b^5 = 521$

9. $672b^4 = 91$

10. $127b^7 = 56$

SECTION 3.2 EXERCISES

In Exercises 1–6, tell whether the function is an exponential growth function or exponential decay function, and find the constant percentage rate of growth or decay.

1. $P(t) = 3.5 \cdot 1.09^t$

2. $P(t) = 4.3 \cdot 1.018^t$

3. $f(x) = 78{,}963 \cdot 0.968^x$

4. $f(x) = 5607 \cdot 0.9968^x$

5. $g(t) = 247 \cdot 2^t$

6. $g(t) = 43 \cdot 0.05^t$

In Exercises 7–18, determine the exponential function that satisfies the given conditions.

7. Initial value $=$ 5, increasing at a rate of 17% per year

8. Initial value $=$ 52, increasing at a rate of 2.3% per day

9. Initial value $=$ 16, decreasing at a rate of 50% per month

10. Initial value $=$ 5, decreasing at a rate of 0.59% per week

11. Initial population $=$ 28,900, decreasing at a rate of 2.6% per year

12. Initial population $=$ 502,000, increasing at a rate of 1.7% per year

13. Initial height $=$ 18 cm, growing at a rate of 5.2% per week

14. Initial mass $=$ 15 g, decreasing at a rate of 4.6% per day

15. Initial mass $=$ 0.6 g, doubling every 3 days

16. Initial population $=$ 250, doubling every 7.5 hours

17. Initial mass $=$ 592 g, halving once every 6 years

18. Initial mass $=$ 17 g, halving once every 32 hours

In Exercises 19 and 20, determine a formula for the exponential function whose values are given in Table 3.11.

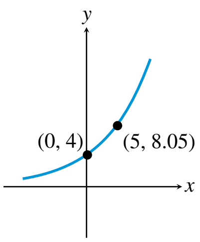
19. $f(x)$ **20.** $g(x)$

Table 3.11 Values for Two Exponential Functions

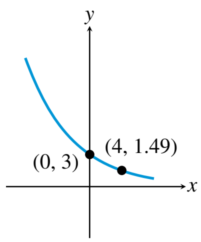
x	$f(x)$	$g(x)$
-2	1.472	-9.0625
-1	1.84	-7.25
0	2.3	-5.8
1	2.875	-4.64
2	3.59375	-3.7123

In Exercises 21 and 22, determine a formula for the exponential function whose graph is shown in the figure.

21.



22.



In Exercises 23–26, find the logistic function that satisfies the given conditions.

23. Initial value $=$ 10, limit to growth $=$ 40, passing through $(1, 20)$.

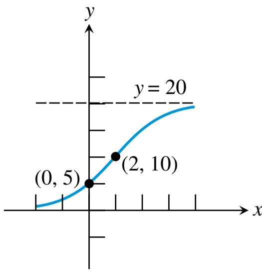
24. Initial value $=$ 12, limit to growth $=$ 60, passing through $(1, 24)$.

25. Initial population $=$ 16, maximum sustainable population $=$ 128, passing through $(5, 32)$.

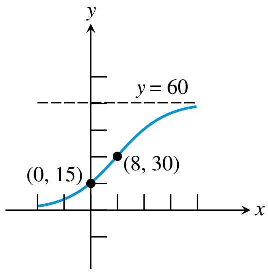
26. Initial height $=$ 5, limit to growth $=$ 30, passing through $(3, 15)$.

In Exercises 27 and 28, determine a formula for the logistic function whose graph is shown in the figure.

27.



28.



29. Exponential Growth The 2000 population of Jacksonville, Florida, was 736,000 and was increasing at the rate of 1.49% each year. At that rate, when will the population be 1 million?

30. Exponential Growth The 2000 population of Las Vegas, Nevada, was 478,000 and is increasing at the rate of 6.28% each year. At that rate, when will the population be 1 million?

31. Exponential Growth The population of Smallville in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940.

(b) Predict when the population reached 50,000.

32. Exponential Growth The population of River City in the year 1910 was 4200. Assume the population increased at a rate of 2.25% per year.

(a) Estimate the population in 1930 and 1945.

(b) Predict when the population reached 20,000.

33. Radioactive Decay The half-life of a certain radioactive substance is 14 days. There are 6.6 g present initially.

(a) Express the amount of substance remaining as a function of time t.

(b) When will there be less than 1 g remaining?

34. Radioactive Decay The half-life of a certain radioactive substance is 65 days. There are 3.5 g present initially.

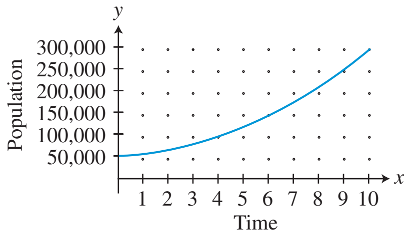
(a) Express the amount of substance remaining as a function of time t.

(b) When will there be less than 1 g remaining?

35. Writing to Learn Without using formulas or graphs, compare and contrast exponential functions and linear functions.

36. Writing to Learn Without using formulas or graphs, compare and contrast exponential functions and logistic functions.

37. Writing to Learn Using the population model that is graphed in the figure, explain why the time it takes the population to double (doubling time) is independent of the population size.



38. Writing to Learn Explain why the half-life of a radioactive substance is independent of the initial amount of the substance that is present.

39. Bacteria Growth The number B of bacteria in a petri dish culture after t hours is given by

$$B = 100e^{0.693t}.$$

When will the number of bacteria be 200? Estimate the doubling time of the bacteria.

40. Radiocarbon Dating The amount C in grams of carbon-14 present in a certain substance after t years is given by

$$C = 20e^{-0.0001216t}.$$

Estimate the half-life of carbon-14.

41. Atmospheric Pressure Determine the atmospheric pressure outside an aircraft flying at 52,800 ft (10 mi above sea level).

42. Atmospheric Pressure Find the altitude above sea level at which the atmospheric pressure is 2.5 lb/in.^{2}.

43. Population Modeling Use the 1950–2000 data in Table 3.12 and exponential regression to predict Los Angeles's population for 2007. Compare the result with the listed value for 2007. [*Hint:* Let 1900 be $t = 0$.]

44. Population Modeling Use the 1950–2000 data in Table 3.12 and exponential regression to predict Phoenix's population for 2007. Compare the result with the listed value for 2007. Repeat these steps using 1960–2000 data to create the model. [*Hint:* Let 1900 be $t = 0$.]

Table 3.12 Populations of Two U.S. Cities (in thousands)

Year	Los Angeles	Phoenix
1950	1970	107
1960	2479	439
1970	2812	584
1980	2969	790
1990	3485	983
2000	3695	1321
2007	3834	1552

Source: World Almanac and Book of Facts 2002, 2009.

45. Spread of Flu The number of students infected with flu at Springfield High School after t days is modeled by the function

$$P(t) = \frac{800}{1 + 49e^{-0.2t}}.$$

(a) What was the initial number of infected students?

(b) When will the number of infected students be 200?

(c) The school will close when 300 of the 800-student body are infected. When will the school close?

46. Population of Deer The population of deer after t years in Cedar State Park is modeled by the function

$$P(t) = \frac{1001}{1 + 90e^{-0.2t}}.$$

(a) What was the initial population of deer?

(b) When will the number of deer be 600?

(c) What is the maximum number of deer possible in the park?

47. Population Growth Using all of the data in Table 3.9, compute a logistic regression model, and use it to predict the U.S. population in 2010.

48. Population Growth Using the data in Table 3.13, confirm the model used in Example 8 of Section 3.1.

Table 3.13 Population of Dallas, Texas

Year	Population
1950	434,462
1960	679,684
1970	844,401
1980	904,599
1990	1,006,877
2000	1,188,589

Source: U.S. Census Bureau.

49. Population Growth Using the data in Table 3.14, confirm the model used in Exercise 56 of Section 3.1.

Table 3.14 Populations of Two U.S. States (in millions)

Year	Arizona	New York
1900	0.1	7.3
1910	0.2	9.1
1920	0.3	10.3
1930	0.4	12.6
1940	0.5	13.5
1950	0.7	14.8
1960	1.3	16.8
1970	1.8	18.2
1980	2.7	17.6
1990	3.7	18.0
2000	5.1	19.0

Source: U.S. Census Bureau.

50. Population Growth Using the data in Table 3.14, compute a logistic regression model for Arizona's population for t years since 1800. Based on your model and the New York population model from Exercise 56 of Section 3.1, will the population of Arizona ever surpass that of New York? If so, when?

Standardized Test Questions

51. True or False Exponential population growth is constrained with a maximum sustainable population. Justify your answer.

52. True or False If the constant percentage rate of an exponential function is negative, then the base of the function is negative. Justify your answer.

In Exercises 53–56, you may use a graphing calculator to solve the problem.

53. Multiple Choice What is the constant percentage growth rate of $P(t) = 1.23 \cdot 1.049^t$?

(A) 49% **(B)** 23% **(C)** 4.9% **(D)** 2.3% **(E)** 1.23%

54. Multiple Choice What is the constant percentage decay rate of $P(t) = 22.7 \cdot 0.834^t$?

(A) 22.7% **(B)** 16.6% **(C)** 8.34%

(D) 2.27% **(E)** 0.834%

55. Multiple Choice A single-cell amoeba doubles every 4 days. About how long will it take one amoeba to produce a population of 1000?

(A) 10 days **(B)** 20 days **(C)** 30 days

(D) 40 days **(E)** 50 days

56. Multiple Choice A rumor spreads logistically so that $S(t) = 789/(1 + 16 \cdot e^{-0.8t})$ models the number of persons who have heard the rumor by the end of t days. Based on this model, which of the following is **true**?

(A) After 0 days, 16 people have heard the rumor.

(B) After 2 days, 439 people have heard the rumor.

(C) After 4 days, 590 people have heard the rumor.

(D) After 6 days, 612 people have heard the rumor.

(E) After 8 days, 769 people have heard the rumor.

Explorations

57. Population Growth **(a)** Use the 1900–1990 data in Table 3.9 and *logistic* regression to predict the U.S. population for 2000.

(b) Writing to Learn Compare the prediction with the value listed in the table for 2000.

(c) Noting the results of Example 6, which model—exponential or logistic—makes the better prediction in this case?

58. Population Growth Use all of the data in Tables 3.9 and 3.15.

(a) Based on exponential growth models, will Mexico's population surpass that of the United States, and if so, when?

(b) Based on logistic growth models, will Mexico's population surpass that of the United States, and if so, when?

(c) What are the maximum sustainable populations for the two countries?

(d) Writing to Learn Which model—exponential or logistic—is more valid in this case? Justify your choice.

Table 3.15 Population of Mexico (in millions)

Year	Population
1900	13.6
1950	25.8
1960	34.9
1970	48.2
1980	66.8
1990	88.1
2001	101.9
2025	130.2
2050	154.0

Sources: 1992 Statesman's Yearbook and World Almanac and Book of Facts 2002.

Extending the Ideas

59. The **hyperbolic sine function** is defined by $\sinh(x) = (e^x - e^{-x})/2$. Prove that sinh is an odd function.

60. The **hyperbolic cosine function** is defined by $\cosh(x) = (e^x + e^{-x})/2$. Prove that cosh is an even function.

61. The **hyperbolic tangent function** is defined by $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$.

(a) Prove that $\tanh(x) = \sinh(x)/\cosh(x)$.

(b) Prove that tanh is an odd function.

(c) Prove that $f(x) = 1 + \tanh(x)$ is a logistic function.