CHAPTER 3

Exponential, Logistic, and Logarithmic Functions

The loudness of a sound we hear is based on the intensity of the associated sound wave. This sound intensity is the energy per unit time of the wave over a given area, measured in watts per square meter (W/m^2). The intensity is greatest near the source and decreases as you move away, whether the sound is rustling leaves or rock music. Because of the wide range of audible sound intensities, they are generally converted into *decibels*, which are based on logarithms. See page 280.

Chapter 3 Overview

In this chapter, we study three interrelated families of functions: exponential, logistic, and logarithmic functions. Polynomial functions, rational functions, and power functions with rational exponents are **algebraic functions**— functions obtained by adding, subtracting, multiplying, and dividing constants and an independent variable, and raising expressions to integer powers and extracting roots. In this chapter and the next one, we explore **transcendental functions**, which go beyond, or transcend, these algebraic operations.

Just like their algebraic cousins, exponential, logistic, and logarithmic functions have wide application. Exponential functions model growth and decay over time, such as *unrestricted* population growth and the decay of radioactive substances. Logistic functions model *restricted* population growth, certain chemical reactions, and the spread of rumors and diseases. Logarithmic functions are the basis of the Richter scale of earthquake intensity, the pH acidity scale, and the decibel measurement of sound.

The chapter closes with a study of the mathematics of finance, an application of exponential and logarithmic functions often used when making investments.

3.1 Exponential and Logistic Functions

What you'll learn about

- Exponential Functions and Their Graphs
- The Natural Base e
- Logistic Functions and Their Graphs
- Population Models

... and why

Exponential and logistic functions model many growth patterns, including the growth of human and animal populations.

Exponential Functions and Their Graphs

The functions $f(x) = x^2$ and $g(x) = 2^x$ each involve a base raised to a power, but the roles are reversed:

- For $f(x) = x^2$, the base is the variable x, and the exponent is the constant 2; f is a familiar monomial and power function.
- For $g(x) = 2^x$, the base is the constant 2, and the exponent is the variable x; g is an *exponential function*. See Figure 3.1.

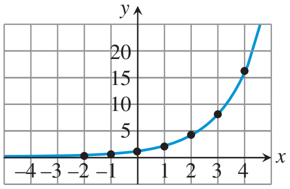


FIGURE 3.1 Sketch of $g(x) = 2^x$.

DEFINITION Exponential Functions

Let a and b be real number constants. An **exponential function** in x is a function that can be written in the form

$$f(x) = a \cdot b^x,$$

where a is nonzero, b is positive, and $b \neq 1$. The constant a is the *initial value* of f (the value at $x = 0$), and b is the **base**.

Exponential functions are defined and continuous for all real numbers. It is important to recognize whether a function is an exponential function.

EXAMPLE 1 Identifying Exponential Functions

(a) $f(x) = 3^x$ is an exponential function, with an initial value of 1 and base of 3.

(b) $g(x) = 6x^{-4}$ is *not* an exponential function because the base x is a variable and the exponent is a constant; g is a power function.

(c) $h(x) = -2 \cdot 1.5^x$ is an exponential function, with an initial value of -2 and base of 1.5.

(d) $k(x) = 7 \cdot 2^{-x}$ is an exponential function, with an initial value of 7 and base of 1/2 because $2^{-x} = (2^{-1})^x = (1/2)^x$.

(e) $q(x) = 5 \cdot 6^{\pi}$ is *not* an exponential function because the exponent π is a constant; q is a constant function. ***Now try Exercise 1.***

One way to evaluate an exponential function, when the inputs are rational numbers, is to use the properties of exponents.

EXAMPLE 2 Computing Exponential Function Values for Rational Number Inputs

For $f(x) = 2^x$,

(a) $f(4) = 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

(b) $f(0) = 2^0 = 1$

(c) $f(-3) = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8} = 0.125$

(d) $f\left(\dfrac{1}{2}\right) = 2^{1/2} = \sqrt{2} = 1.4142\ldots$

(e) $f\left(-\dfrac{3}{2}\right) = 2^{-3/2} = \dfrac{1}{2^{3/2}} = \dfrac{1}{\sqrt{2^3}} = \dfrac{1}{\sqrt{8}} = 0.35355\ldots$

Now try Exercise 7.

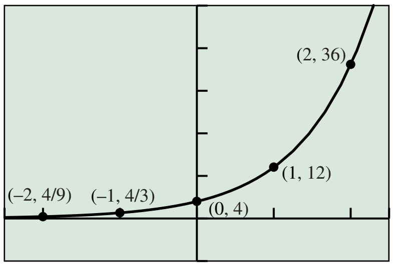
There is no way to use properties of exponents to express an exponential function's value for *irrational* inputs. For example, if $f(x) = 2^x$, $f(\pi) = 2^{\pi}$, but what does 2^{π} mean? Using properties of exponents, $2^3 = 2 \cdot 2 \cdot 2$, $2^{3.1} = 2^{31/10} = \sqrt[10]{2^{31}}$. So we can find meaning for 2^{π} by using successively closer *rational* approximations to π as shown in Table 3.1.

Table 3.1 Values of $f(x) = 2^x$ for Rational Numbers x Approaching $\pi = 3.14159265\ldots$

x	3	3.1	3.14	3.141	3.1415	3.14159
2^x	8	8.5. . .	8.81. . .	8.821. . .	8.8244. . .	8.82496. . .

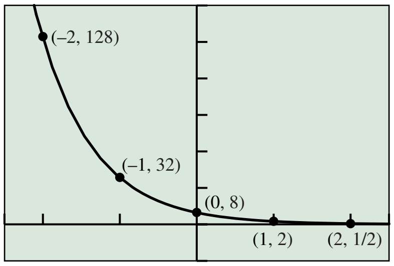
We can conclude that $f(\pi) = 2^{\pi} \approx 8.82$, which could be found directly using a grapher. The methods of calculus permit a more rigorous definition of exponential functions than we give here, a definition that allows for both rational and irrational inputs.

The way exponential functions change makes them useful in applications. This pattern of change can best be observed in tabular form.



[–2.5, 2.5] by [–10, 50]

(a)



[–2.5, 2.5] by [–25, 150]

(b)

FIGURE 3.2 Graphs of (a) $g(x) = 4 \cdot 3^x$ and (b) $h(x) = 8 \cdot (1/4)^x$. (Example 3)

EXAMPLE 3 Finding an Exponential Function from Its Table of Values

Determine formulas for the exponential functions g and h whose values are given in Table 3.2.

Table 3.2 Values for Two Exponential Functions

x	$g(x)$		$h(x)$	
−2	4/9		128	
		$\times 3$		$\times \frac{1}{4}$
−1	4/3		32	
		$\times 3$		$\times \frac{1}{4}$
0	4		8	
		$\times 3$		$\times \frac{1}{4}$
1	12		2	
		$\times 3$		$\times \frac{1}{4}$
2	36		1/2	

SOLUTION Because g is exponential, $g(x) = a \cdot b^x$. Because $g(0) = 4$, the initial value a is 4. Because $g(1) = 4 \cdot b^1 = 12$, the base b is 3. So,

$$g(x) = 4 \cdot 3^x.$$

Because h is exponential, $h(x) = a \cdot b^x$. Because $h(0) = 8$, the initial value a is 8. Because $h(1) = 8 \cdot b^1 = 2$, the base b is 1/4. So,

$$h(x) = 8 \cdot \left(\frac{1}{4}\right)^x.$$

Figure 3.2 shows the graphs of these functions pass through the points whose coordinates are given in Table 3.2. ***Now try Exercise 11.***

Table 3.3 Values for a General Exponential Function $f(x) = a \cdot b^x$

x	$a \cdot b^x$	
−2	ab^{-2}	
		$\times b$
−1	ab^{-1}	
		$\times b$
0	a	
		$\times b$
1	ab	
		$\times b$
2	ab^2	

Observe the patterns in the $g(x)$ and $h(x)$ columns of Table 3.2. The $g(x)$ values increase by a factor of 3 and the $h(x)$ values decrease by a factor of 1/4, as we add 1 to x moving from one row of the table to the next. In each case, the change factor is the base of the exponential function. This pattern generalizes to all exponential functions as illustrated in Table 3.3.

In Table 3.3, as x increases by 1, the function value is multiplied by the base b. This relationship leads to the following *recursive formula*.

Exponential Growth and Decay

For any exponential function $f(x) = a \cdot b^x$ and any real number x,

$$f(x + 1) = b \cdot f(x).$$

If $a > 0$ and $b > 1$, the function f is increasing and is an **exponential growth function**. The base b is its **growth factor**.

If $a > 0$ and $b < 1$, f is decreasing and is an **exponential decay function**. The base b is its **decay factor**.

In Example 3, g is an exponential growth function, and h is an exponential decay function. As x increases by 1, $g(x) = 4 \cdot 3^x$ grows by a factor of 3, and $h(x) = 8 \cdot (1/4)^x$ decays by a factor of 1/4. Whenever the initial value is positive, the base of an exponential function, like the slope of a linear function, tells us whether the function is increasing or decreasing and by how much.

So far, we have focused most of our attention on the algebraic and numerical aspects of exponential functions. We now turn our attention to the graphs of these functions.

EXPLORATION 1 Graphs of Exponential Functions

1. Graph each function in the viewing window $[-2, 2]$ by $[-1, 6]$.

   (a) $y_1 = 2^x$ **(b)** $y_2 = 3^x$ **(c)** $y_3 = 4^x$ **(d)** $y_4 = 5^x$

   - Which point is common to all four graphs?
   - Analyze the functions for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

2. Graph each function in the viewing window $[-2, 2]$ by $[-1, 6]$.

   (a) $y_1 = \left(\frac{1}{2}\right)^x$ **(b)** $y_2 = \left(\frac{1}{3}\right)^x$

   (c) $y_3 = \left(\frac{1}{4}\right)^x$ **(d)** $y_4 = \left(\frac{1}{5}\right)^x$

   - Which point is common to all four graphs?
   - Analyze the functions for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

We summarize what we have learned about exponential functions with an initial value of 1.

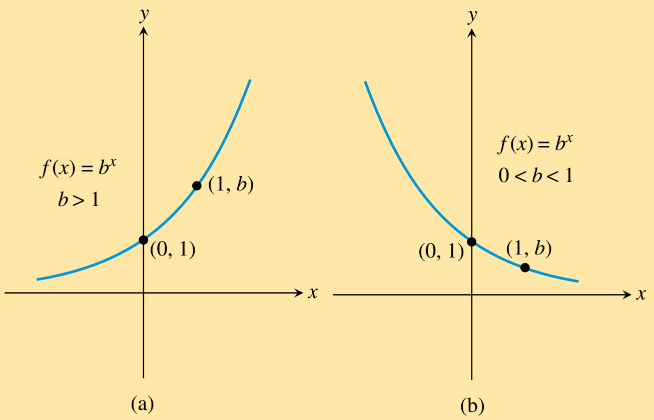


FIGURE 3.3 Graphs of $f(x) = b^x$ for (a) $b > 1$ and (b) $0 < b < 1$.

Exponential Functions $f(x) = b^x$

Domain: All reals
Range: $(0, \infty)$
Continuous
No symmetry: neither even nor odd
Bounded below, but not above
No local extrema
Horizontal asymptote: $y = 0$
No vertical asymptotes

If $b > 1$ (see Figure 3.3a), then

- f is an increasing function,
- $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$.

If $0 < b < 1$ (see Figure 3.3b), then

- f is a decreasing function,
- $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = 0$.

The translations, reflections, stretches, and shrinks studied in Section 1.5, together with our knowledge of the graphs of basic exponential functions, allow us to predict the graphs of the functions in Example 4.

EXAMPLE 4 Transforming Exponential Functions

Describe how to transform the graph of $f(x) = 2^x$ into the graph of the given function. Sketch the graphs by hand and support your answer with a grapher.

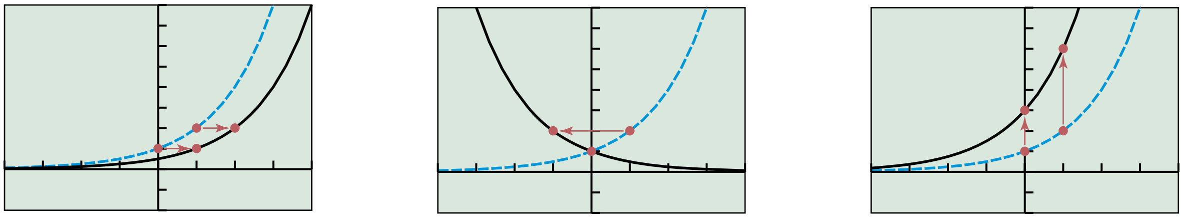
(a) $g(x) = 2^{x-1}$ **(b)** $h(x) = 2^{-x}$ **(c)** $k(x) = 3 \cdot 2^x$

SOLUTION

(a) The graph of $g(x) = 2^{x-1}$ is obtained by translating the graph of $f(x) = 2^x$ by 1 unit to the right (Figure 3.4a).

(b) We can obtain the graph of $h(x) = 2^{-x}$ by reflecting the graph of $f(x) = 2^x$ across the y-axis (Figure 3.4b). Because $2^{-x} = (2^{-1})^x = (1/2)^x$, we can also think of h as an exponential function with an initial value of 1 and a base of 1/2.

(c) We can obtain the graph of $k(x) = 3 \cdot 2^x$ by vertically stretching the graph of $f(x) = 2^x$ by a factor of 3 (Figure 3.4c). *Now try Exercise 15.*



[–4, 4] by [–2, 8]
(a)

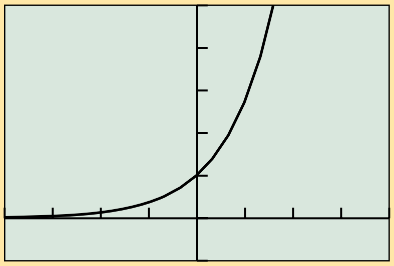
[–4, 4] by [–2, 8]
(b)

[–4, 4] by [–2, 8]
(c)

FIGURE 3.4 The graph of $f(x) = 2^x$ shown with (a) $g(x) = 2^{x-1}$, (b) $h(x) = 2^{-x}$, and (c) $k(x) = 3 \cdot 2^x$. (Example 4)

The Natural Base e

The function $f(x) = e^x$ is one of the basic functions introduced in Section 1.3, and is an exponential growth function.



[–4, 4] by [–1, 5]

FIGURE 3.5 The graph of $f(x) = e^x$.

BASIC FUNCTION The Exponential Function

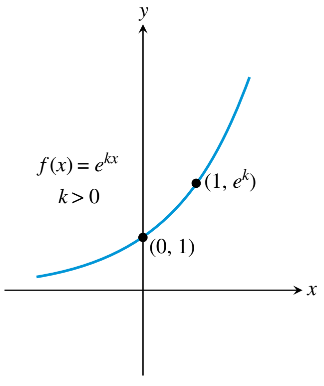
$f(x) = e^x$
Domain: All reals
Range: $(0, \infty)$
Continuous
Increasing for all x
No symmetry
Bounded below, but not above
No local extrema
Horizontal asymptote: $y = 0$
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

Because $f(x) = e^x$ is increasing, it is an exponential growth function, so $e > 1$. But what is e, and what makes this exponential function *the* exponential function?

The letter e is the initial of the last name of Leonhard Euler (1707–1783), who introduced the notation. Because $f(x) = e^x$ has special calculus properties that simplify many calculations, e is the *natural base* of exponential functions for calculus purposes, and $f(x) = e^x$ is considered the *natural exponential function*.

DEFINITION The Natural Base e

$$e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$$



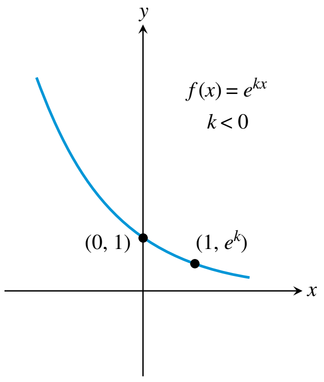


FIGURE 3.6 Graphs of $f(x) = e^{kx}$ for (a) $k > 0$ and (b) $k < 0$.

We cannot compute the irrational number e directly, but using this definition we can obtain successively closer approximations to e, as shown in Table 3.4. Continuing the process in Table 3.4 with a sufficiently accurate computer can show that $e \approx 2.718281828459$.

Table 3.4 Approximations Approaching the Natural Base e

x	1	10	100	1000	10,000	100,000
$(1 + 1/x)^x$	2	2.5. . .	2.70. . .	2.716. . .	2.7181. . .	2.71826. . .

We are usually more interested in the exponential function $f(x) = e^x$ and variations of this function than in the irrational number e. In fact, *any* exponential function can be expressed in terms of the natural base e.

THEOREM Exponential Functions and the Base e

Any exponential function $f(x) = a \cdot b^x$ can be rewritten as

$$f(x) = a \cdot e^{kx},$$

for an appropriately chosen real number constant k.

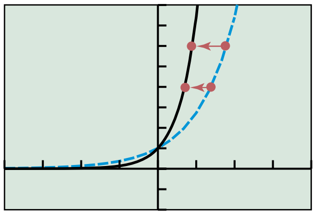
If $a > 0$ and $k > 0$, $f(x) = a \cdot e^{kx}$ is an exponential growth function. (See Figure 3.6a.)

If $a > 0$ and $k < 0$, $f(x) = a \cdot e^{kx}$ is an exponential decay function. (See Figure 3.6b.)

In Section 3.3 we will develop some mathematics so that, for any positive number $b \neq 1$, we can easily find the value of k such that $e^{kx} = b^x$. In the meantime, we can use graphical and numerical methods to approximate k, as you will discover in Exploration 2.

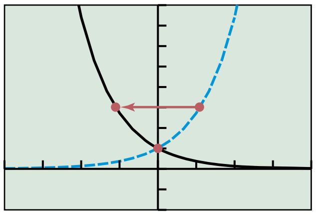
EXPLORATION 2 Choosing k so that $e^{kx} = 2^x$

1. Graph $f(x) = 2^x$ in the viewing window $[-4, 4]$ by $[-2, 8]$.
2. One at a time, overlay the graphs of $g(x) = e^{kx}$ for $k = 0.4, 0.5, 0.6, 0.7$, and 0.8. For which of these values of k does the graph of g most closely match the graph of f?
3. Using tables, find the 3-decimal-place value of k for which the values of g most closely approximate the values of f.



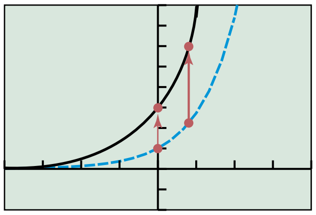
[–4, 4] by [–2, 8]

(a)



[–4, 4] by [–2, 8]

(b)



[–4, 4] by [–2, 8]

(c)

FIGURE 3.7 The graph of $f(x) = e^x$ shown with (a) $g(x) = e^{2x}$, (b) $h(x) = e^{-x}$, and (c) $k(x) = 3e^x$. (Example 5)

EXAMPLE 5 Transforming Exponential Functions

Describe how to transform the graph of $f(x) = e^x$ into the graph of the given function. Sketch the graphs by hand and support your answer with a grapher.

(a) $g(x) = e^{2x}$ **(b)** $h(x) = e^{-x}$ **(c)** $k(x) = 3e^x$

SOLUTION

(a) The graph of $g(x) = e^{2x}$ is obtained by horizontally shrinking the graph of $f(x) = e^x$ by a factor of 2 (Figure 3.7a).

(b) We can obtain the graph of $h(x) = e^{-x}$ by reflecting the graph of $f(x) = e^x$ across the y-axis (Figure 3.7b).

(c) We can obtain the graph of $k(x) = 3e^x$ by vertically stretching the graph of $f(x) = e^x$ by a factor of 3 (Figure 3.7c). ***Now try Exercise 21.***

Logistic Functions and Their Graphs

Exponential growth is *unrestricted*. An exponential growth function increases at an ever-increasing rate and is not bounded above. In many growth situations, however, there is a limit to the possible growth. A plant can only grow so tall. The number of goldfish in an aquarium is limited by the size of the aquarium. In such situations the growth often begins in an exponential manner, but the growth eventually slows and the graph levels out. The associated growth function is bounded both below and above by horizontal asymptotes.

DEFINITION Logistic Growth Functions

Let a, b, c, and k be positive constants, with $b < 1$. A **logistic growth function** in x is a function that can be written in the form

$$f(x) = \frac{c}{1 + a \cdot b^x} \text{ or } f(x) = \frac{c}{1 + a \cdot e^{-kx}}$$

where the constant c is the **limit to growth**.

Aliases for Logistic Growth

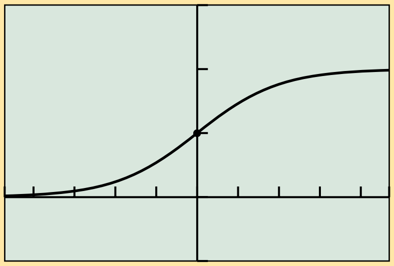
Logistic growth is also known as *restricted, inhibited,* or *constrained exponential growth.*

If $b > 1$ or $k < 0$, these formulas yield **logistic decay functions**. Unless otherwise stated, all *logistic functions* in this book will be logistic growth functions.

By setting $a = c = k = 1$, we obtain the **logistic function**

$$f(x) = \frac{1}{1 + e^{-x}}.$$

This function, though related to the exponential function e^x, *cannot* be obtained from e^x by translations, reflections, and horizontal and vertical stretches and shrinks. So we give the logistic function a formal introduction:



[–4.7, 4.7] by [–0.5, 1.5]

FIGURE 3.8 The graph of $f(x) = 1/(1 + e^{-x})$.

BASIC FUNCTION The Logistic Function

$$f(x) = \frac{1}{1 + e^{-x}}$$

Domain: All reals
Range: $(0, 1)$
Continuous
Increasing for all x
Symmetric about $(0, 1/2)$, but neither even nor odd
Bounded below and above
No local extrema
Horizontal asymptotes: $y = 0$ and $y = 1$
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 1$

All logistic growth functions have graphs much like the basic logistic function. Their end behavior is always described by the equations

$$\lim_{x \to -\infty} f(x) = 0 \text{ and } \lim_{x \to \infty} f(x) = c,$$

where c is the limit to growth (see Exercise 73). All logistic functions are bounded by their horizontal asymptotes, $y = 0$ and $y = c$, and have a range of $(0, c)$. Although every logistic function is symmetric about the point of its graph with y- coordinate $c/2$, this point of symmetry is usually not the y-intercept, as we can see in Example 6.

EXAMPLE 6 Graphing Logistic Growth Functions

Graph the function. Find the y-intercept and the horizontal asymptotes.

(a) $f(x) = \dfrac{8}{1 + 3 \cdot 0.7^x}$ **(b)** $g(x) = \dfrac{20}{1 + 2e^{-3x}}$

SOLUTION

(a) The graph of $f(x) = 8/(1 + 3 \cdot 0.7^x)$ is shown in Figure 3.9a. The y-intercept is

$$f(0) = \frac{8}{1 + 3 \cdot 0.7^0} = \frac{8}{1 + 3} = 2.$$

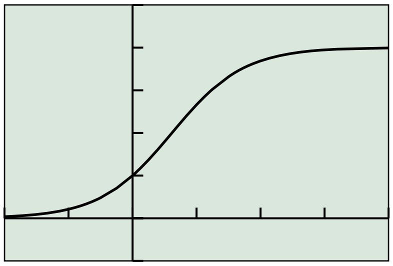
Because the limit to growth is 8, the horizontal asymptotes are $y = 0$ and $y = 8$.

(b) The graph of $g(x) = 20/(1 + 2e^{-3x})$ is shown in Figure 3.9b. The y-intercept is

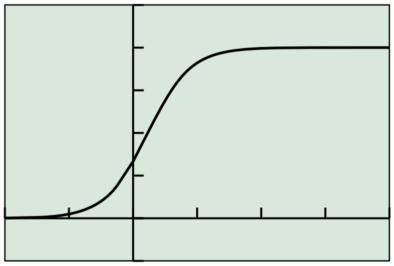
$$g(0) = \frac{20}{1 + 2e^{-3 \cdot 0}} = \frac{20}{1 + 2} = 20/3 \approx 6.67.$$

Because the limit to growth is 20, the horizontal asymptotes are $y = 0$ and $y = 20$.

Now try Exercise 41.



[–10, 20] by [–2, 10]
(a)



[–2, 4] by [–5, 25]
(b)

FIGURE 3.9 The graphs of (a) $f(x) = 8/(1 + 3 \cdot 0.7^x)$ and (b) $g(x) = 20/(1 + 2e^{-3x})$. (Example 6)

Population Models

Exponential and logistic functions have many applications. One area where both types of functions are used is in modeling population. Between 1990 and 2000, both Phoenix and San Antonio passed the 1 million mark. With its Silicon Valley industries, San Jose, California, appears to be the next U.S. city destined to surpass 1 million residents. When a city's population is growing rapidly, as in the case of San Jose, exponential growth is a reasonable model.

EXAMPLE 7 Modeling San Jose's Population

Using the data in Table 3.5 and assuming the growth is exponential, when will the population of San Jose, California, surpass 1 million persons?

SOLUTION

Model Let $P(t)$ be the population of San Jose t years after July 1, 2000. (See margin note.) Because P is exponential, $P(t) = P_0 \cdot b^t$, where P_0 is the initial (2000) population of 898,759. From Table 3.5 we see that $P(7) = 898759b^7 = 939899$. So,

$$b = \sqrt[7]{\frac{939{,}899}{898{,}759}} \approx 1.0064$$

and $P(t) = 898{,}759 \cdot 1.0064^t$.

Solve Graphically Figure 3.10 shows that this population model intersects $y = 1{,}000{,}000$ when the independent variable is about 16.73.

Interpret Because 16.73 yr after mid-2000 is in the first half of 2017, according to this model the population of San Jose will surpass the 1 million mark in early 2017.

Now try Exercise 51.

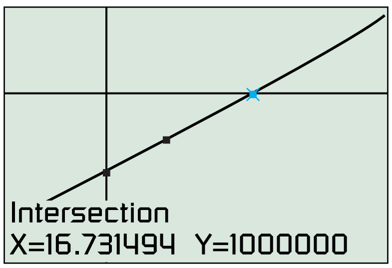
A Note on Population Data

When the U.S. Census Bureau reports a population estimate for a given year, it generally represents the population at the middle of the year, or July 1. We will assume this to be the case when interpreting our results to population problems unless otherwise noted.

Table 3.5 The Population of San Jose, California

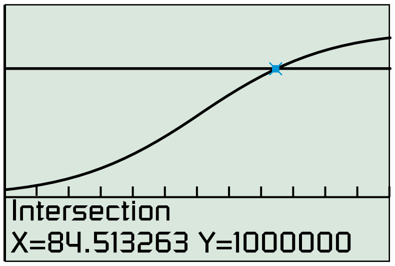
Year	Population
2000	898,759
2007	939,899

Source: U.S. Census Bureau.



[–10, 30] by [800 000, 1 100 000]

FIGURE 3.10 A population model for San Jose, California. (Example 7)



[0, 120] by [−500 000, 1 500 000]

FIGURE 3.11 A population model for Dallas, Texas. (Example 8)

While San Jose's population is soaring, in other major cities, such as Dallas, the population growth is slowing. The once sprawling Dallas is now *constrained* by its neighboring cities. *A logistic function is often an appropriate model for restricted growth*, such as the growth that Dallas is experiencing.

EXAMPLE 8 Modeling Dallas's Population

Based on recent census data, a logistic model for the population of Dallas, t years after 1900, is as follows:

$$P(t) = \frac{1{,}301{,}642}{1 + 21.602e^{-0.05054t}}$$

According to this model, when was the population 1 million?

SOLUTION Figure 3.11 shows that the population model intersects $y = 1{,}000{,}000$ when the independent variable is about 84.51. Because 84.51 yr after mid-1900 is at the beginning of 1985, if Dallas's population has followed this logistic model, its population was 1 million then. *Now try Exercise 55.*

QUICK REVIEW 3.1 *(For help, go to Sections A.1 and P.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\sqrt[3]{-216}$ **2.** $\sqrt[3]{\frac{125}{8}}$

3. $27^{2/3}$ **4.** $4^{5/2}$

In Exercises 5–8, rewrite the expression using a single positive exponent.

5. $(2^{-3})^4$ **6.** $(3^4)^{-2}$

7. $(a^{-2})^3$ **8.** $(b^{-3})^{-5}$

In Exercises 9–10, use a calculator to evaluate the expression.

9. $\sqrt[5]{-5.37824}$ **10.** $\sqrt[4]{92.3521}$

SECTION 3.1 EXERCISES

In Exercises 1–6, which of the following are exponential functions? For those that are exponential functions, state the initial value and the base. For those that are not, explain why not.

1. $y = x^8$

2. $y = 3^x$

3. $y = 5^x$

4. $y = 4^2$

5. $y = x^{\sqrt{x}}$

6. $y = x^{1.3}$

In Exercises 7–10, compute the exact value of the function for the given x-value without using a calculator.

7. $f(x) = 3 \cdot 5^x$ for $x = 0$

8. $f(x) = 6 \cdot 3^x$ for $x = -2$

9. $f(x) = -2 \cdot 3^x$ for $x = 1/3$

10. $f(x) = 8 \cdot 4^x$ for $x = -3/2$

In Exercises 11 and 12, determine a formula for the exponential function whose values are given in Table 3.6.

11. $f(x)$

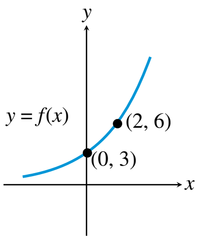
12. $g(x)$

Table 3.6 Values for Two Exponential Functions

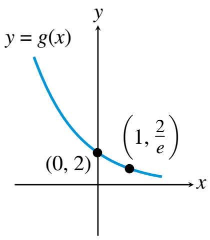
x	$f(x)$	$g(x)$
−2	6	108
−1	3	36
0	3/2	12
1	3/4	4
2	3/8	4/3

In Exercises 13 and 14, determine a formula for the exponential function whose graph is shown in the figure.

13. $f(x)$



14. $g(x)$



In Exercises 15–24, describe how to transform the graph of f into the graph of g. Sketch the graphs by hand and support your answer with a grapher.

15. $f(x) = 2^x,\ g(x) = 2^{x-3}$

16. $f(x) = 3^x,\ g(x) = 3^{x+4}$

17. $f(x) = 4^x,\ g(x) = 4^{-x}$

18. $f(x) = 2^x,\ g(x) = 2^{5-x}$

19. $f(x) = 0.5^x,\ g(x) = 3 \cdot 0.5^x + 4$

20. $f(x) = 0.6^x,\ g(x) = 2 \cdot 0.6^{3x}$

21. $f(x) = e^x,\ g(x) = e^{-2x}$

22. $f(x) = e^x,\ g(x) = -e^{-3x}$

23. $f(x) = e^x,\ g(x) = 2e^{3-3x}$

24. $f(x) = e^x,\ g(x) = 3e^{2x} - 1$

In Exercises 25–30, **(a)** match the given function with its graph. **(b) Writing to Learn** Explain how to make the choice without using a grapher.

25. $y = 3^x$

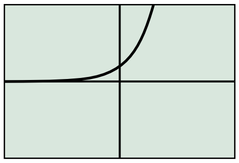
26. $y = 2^{-x}$

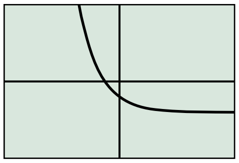
27. $y = -2^x$

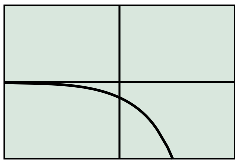
28. $y = -0.5^x$

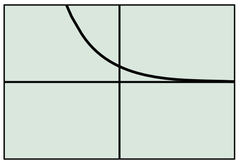
29. $y = 3^{-x} - 2$

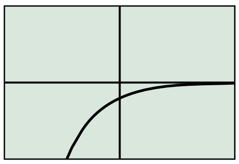
30. $y = 1.5^x - 2$

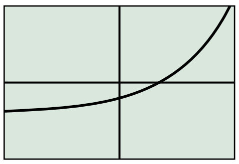

(a)


(b)


(c)


(d)


(e)


(f)

In Exercises 31–34, state whether the function is an exponential growth function or exponential decay function, and describe its end behavior using limits.

31. $f(x) = 3^{-2x}$

32. $f(x) = \left(\frac{1}{e}\right)^x$

33. $f(x) = 0.5^x$

34. $f(x) = 0.75^{-x}$

In Exercises 35–38, solve the inequality graphically.

35. $9^x < 4^x$

36. $6^{-x} > 8^{-x}$

37. $\left(\frac{1}{4}\right)^x > \left(\frac{1}{3}\right)^x$

38. $\left(\frac{1}{3}\right)^x < \left(\frac{1}{2}\right)^x$

Group Activity In Exercises 39 and 40, use the properties of exponents to prove that two of the given three exponential functions are identical. Support graphically.

39. **(a)** $y_1 = 3^{2x+4}$

(b) $y_2 = 3^{2x} + 4$

(c) $y_3 = 9^{x+2}$

40. **(a)** $y_1 = 4^{3x-2}$

(b) $y_2 = 2(2^{3x-2})$

(c) $y_3 = 2^{3x-1}$

In Exercises 41–44, use a grapher to graph the function. Find the y-intercept and the horizontal asymptotes.

41. $f(x) = \dfrac{12}{1 + 2 \cdot 0.8^x}$

42. $f(x) = \dfrac{18}{1 + 5 \cdot 0.2^x}$

43. $f(x) = \dfrac{16}{1 + 3e^{-2x}}$

44. $g(x) = \dfrac{9}{1 + 2e^{-x}}$

In Exercises 45–50, graph the function and analyze it for domain, range, continuity, increasing or decreasing behavior, symmetry, boundedness, extrema, asymptotes, and end behavior.

45. $f(x) = 3 \cdot 2^x$

46. $f(x) = 4 \cdot 0.5^x$

47. $f(x) = 4 \cdot e^{3x}$

48. $f(x) = 5 \cdot e^{-x}$

49. $f(x) = \dfrac{5}{1 + 4 \cdot e^{-2x}}$

50. $f(x) = \dfrac{6}{1 + 2 \cdot e^{-x}}$

51. Population Growth Using the midyear data in Table 3.7 and assuming the growth is exponential, when did the population of Austin surpass 800,000 persons?

Table 3.7 Populations of Two Major U.S. Cities

City	1990 Population	2000 Population
Austin, Texas	465,622	656,562
Columbus, Ohio	632,910	711,265

Source: World Almanac and Book of Facts 2005.

52. Population Growth Using the data in Table 3.7 and assuming the growth is exponential, when would the population of Columbus surpass 800,000 persons?

53. Population Growth Using the data in Table 3.7 and assuming the growth is exponential, when were the populations of Austin and Columbus equal?

54. Population Growth Using the data in Table 3.7 and assuming the growth is exponential, which city—Austin or Columbus—would reach a population of 1 million first, and in what year?

55. Population Growth Using 20th-century U.S. census data, the population of Ohio can be modeled by

$$P(t) = \frac{12.79}{1 + 2.402e^{-0.0309x}},$$

where P is the population in millions and t is the number of years since April 1, 1900. Based on this model, when was the population of Ohio 10 million?

56. Population Growth Using 20th-century U.S. census data, the population of New York state can be modeled by

$$P(t) = \frac{19.875}{1 + 57.993e^{-0.035005t}},$$

where P is the population in millions and t is the number of years since 1800. Based on this model,

(a) What was the population of New York in 1850?

(b) What will New York state's population be in 2015?

(c) What is New York's *maximum sustainable population* (limit to growth)?

57. Bacteria Growth The number B of bacteria in a petri dish culture after t hours is given by

$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present?

(b) How many bacteria are present after 6 hours?

58. Carbon Dating The amount C in grams of carbon-14 present in a certain substance after t years is given by

$$C = 20e^{-0.0001216t}.$$

(a) What was the initial amount of carbon-14 present?

(b) How much is left after 10,400 years? When will the amount left be 10 g?

Standardized Test Questions

59. True or False Every exponential function is strictly increasing. Justify your answer.

60. True or False Every logistic growth function has two horizontal asymptotes. Justify your answer.

In Exercises 61–64, solve the problem without using a calculator.

61. Multiple Choice Which of the following functions is exponential?

(A) $f(x) = a^2$ **(B)** $f(x) = x^3$

(C) $f(x) = x^{2/3}$ **(D)** $f(x) = \sqrt[3]{x}$

(E) $f(x) = 8^x$

62. Multiple Choice What point do all functions of the form $f(x) = b^x (b > 0)$ have in common?

(A) $(1, 1)$ **(B)** $(1, 0)$ **(C)** $(0, 1)$

(D) $(0, 0)$ **(E)** $(-1, -1)$

63. Multiple Choice The growth factor for $f(x) = 4 \cdot 3^x$ is

(A) 3. **(B)** 4. **(C)** 12.

(D) 64. **(E)** 81.

64. Multiple Choice For $x > 0$, which of the following is **true?**

(A) $3^x > 4^x$ **(B)** $7^x > 5^x$ **(C)** $(1/6)^x > (1/2)^x$

(D) $9^{-x} > 8^{-x}$ **(E)** $0.17^x > 0.32^x$

Explorations

65. Graph each function and analyze it for domain, range, increasing or decreasing behavior, boundedness, extrema, asymptotes, and end behavior.

(a) $f(x) = x \cdot e^x$ **(b)** $g(x) = \dfrac{e^{-x}}{x}$

66. Use the properties of exponents to solve each equation. Support graphically.

(a) $2^x = 4^2$ **(b)** $3^x = 27$

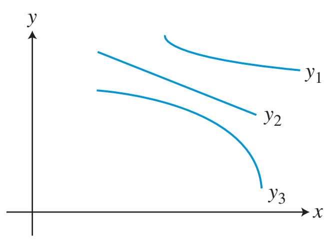
(c) $8^{x/2} = 4^{x+1}$ **(d)** $9^x = 3^{x+1}$

Extending the Ideas

67. Writing to Learn Table 3.8 gives function values for $y = f(x)$ and $y = g(x)$. Also, three different graphs are shown.

Table 3.8 Data for Two Functions

x	$f(x)$	$g(x)$
1.0	5.50	7.40
1.5	5.35	6.97
2.0	5.25	6.44
2.5	5.17	5.76
3.0	5.13	4.90
3.5	5.09	3.82
4.0	5.06	2.44
4.5	5.05	0.71



(a) Which curve of those shown in the graph most closely resembles the graph of $y = f(x)$? Explain your choice.

(b) Which curve most closely resembles the graph of $y = g(x)$? Explain your choice.

68. Writing to Learn Let $f(x) = 2^x$. Explain why the graph of $f(ax + b)$ can be obtained by applying one transformation to the graph of $y = c^x$ for an appropriate value of c. What is c?

Exercises 69–72 refer to the expression $f(a, b, c) = a \cdot b^c$. For example, if $a = 2$, $b = 3$, and $c = x$, the expression is $f(2, 3, x) = 2 \cdot 3^x$, an exponential function.

69. If $b = x$, state conditions on a and c under which the expression $f(a, b, c)$ is a quadratic power function.

70. If $b = x$, state conditions on a and c under which the expression $f(a, b, c)$ is a decreasing linear function.

71. If $c = x$, state conditions on a and b under which the expression $f(a, b, c)$ is an increasing exponential function.

72. If $c = x$, state conditions on a and b under which the expression $f(a, b, c)$ is a decreasing exponential function.

73. Prove that $\lim_{x\to-\infty} \frac{c}{1 + a \cdot b^x} = 0$ and $\lim_{x\to\infty} \frac{c}{1 + a \cdot b^x} = c$, for constants a, b, and c, with $a > 0$, $0 < b < 1$, and $c > 0$.