2.2 Power Functions with Modeling

What you'll learn about

- Power Functions and Variation
- Monomial Functions and Their Graphs
- Graphs of Power Functions
- Modeling with Power Functions

… and why

Power functions specify the proportional relationships of geometry, chemistry, and physics.

Power Functions and Variation

Five of the basic functions introduced in Section 1.3 were power functions. Power functions are an important family of functions in their own right and are important building blocks for other functions.

DEFINITION Power Function

Any function that can be written in the form

$$f(x) = k \cdot x^a, \text{ where } k \text{ and } a \text{ are nonzero constants,}$$

is a **power function**. The constant a is the **power**, and k is the **constant of variation**, or **constant of proportion**. We say $f(x)$ **varies as** the ath power of x, or $f(x)$ **is proportional to** the ath power of x.

In general, if $y = f(x)$ varies as a constant power of x, then y is a power function of x. Many of the most common formulas from geometry and science are power functions.

Name	Formula	Power	Constant of Variation
Circumference	$C = 2\pi r$	1	2π
Area of a circle	$A = \pi r^2$	2	π
Force of gravity	$F = k/d^2$	-2	k
Boyle's Law	$V = k/P$	-1	k

These four power functions involve relationships that can be expressed in the language of *variation and proportion*:

- The circumference of a circle varies directly as its radius.
- The area enclosed by a circle is directly proportional to the square of its radius.
- The force of gravity acting on an object is inversely proportional to the square of the distance from the object to the center of the Earth.
- Boyle's Law states that the volume of an enclosed gas (at a constant temperature) varies inversely as the applied pressure.

The power function formulas with positive powers are statements of **direct variation**, and power function formulas with negative powers are statements of **inverse variation**. Unless the word *inversely* is included in a variation statement, the variation is assumed to be direct, as in Example 1.

EXAMPLE 1 Writing a Power Function Formula

From empirical evidence and the laws of physics it has been found that the period of time T for the full swing of a pendulum varies as the square root of the pendulum's length l, provided that the swing is small relative to the length of the pendulum. Express this relationship as a power function.

SOLUTION Because it does not state otherwise, the variation is direct. So the power is positive. The wording tells us that T is a function of l. Using k as the constant of variation gives us

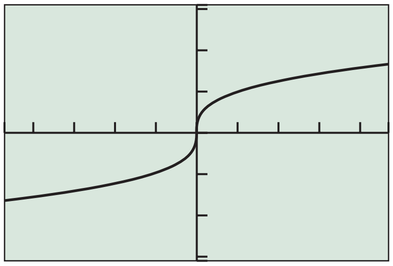
$$T(l) = k\sqrt{l} = k \cdot l^{1/2}.$$

Now try Exercise 17.

Section 1.3 introduced five basic power functions:

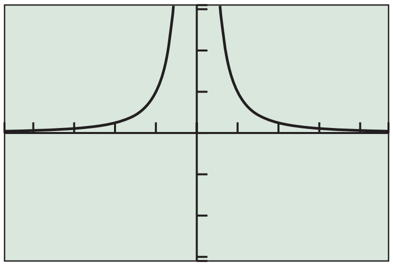
$$x, x^2, x^3, x^{-1} = \frac{1}{x}, \quad \text{and} \quad x^{1/2} = \sqrt{x}$$

Example 2 describes two other power functions: the *cube root function* and the *inverse-square* function.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(a)



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(b)

FIGURE 2.10 The graphs of (a) $f(x) = \sqrt[3]{x} = x^{1/3}$ and (b) $g(x) = 1/x^2 = x^{-2}$. (Example 2)

EXAMPLE 2 Analyzing Power Functions

State the power and constant of variation for the function, graph it, and analyze it.

(a) $f(x) = \sqrt[3]{x}$ **(b)** $g(x) = \dfrac{1}{x^2}$

SOLUTION

(a) Because $f(x) = \sqrt[3]{x} = x^{1/3} = 1 \cdot x^{1/3}$, its power is 1/3, and its constant of variation is 1. The graph of f is shown in Figure 2.10a.

Domain: All reals
Range: All reals
Continuous
Increasing for all x
Symmetric with respect to the origin (an odd function)
Not bounded above or below
No local extrema
No asymptotes
End behavior: $\lim_{x\to-\infty} \sqrt[3]{x} = -\infty$ and $\lim_{x\to\infty} \sqrt[3]{x} = \infty$
Interesting fact: The cube root function $f(x) = \sqrt[3]{x}$ is the inverse of the cubing function.

(b) Because $g(x) = 1/x^2 = x^{-2} = 1 \cdot x^{-2}$, its power is -2, and its constant of variation is 1. The graph of g is shown in Figure 2.10b.

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$
Continuous on its domain; discontinuous at $x = 0$
Increasing on $(-\infty, 0)$; decreasing on $(0, \infty)$
Symmetric with respect to the y-axis (an even function)
Bounded below, but not above
No local extrema

Horizontal asymptote: $y = 0$; vertical asymptote: $x = 0$
End behavior: $\lim_{x\to-\infty} (1/x^2) = 0$ and $\lim_{x\to\infty} (1/x^2) = 0$

Interesting fact: $g(x) = 1/x^2$ is the basis of scientific *inverse-square laws*, for example, the inverse-square gravitational principle $F = k/d^2$ mentioned above.

So $g(x) = 1/x^2$ is sometimes called the inverse-square function, but it is *not* the inverse of the squaring function but rather its *multiplicative* inverse.

Now try Exercise 27.

Monomial Functions and Their Graphs

A single-term polynomial function is a *monomial function*.

DEFINITION Monomial Function

Any function that can be written as

$$f(x) = k \text{ or } f(x) = k \cdot x^n,$$

where k is a constant and n is a positive integer, is a **monomial function**.

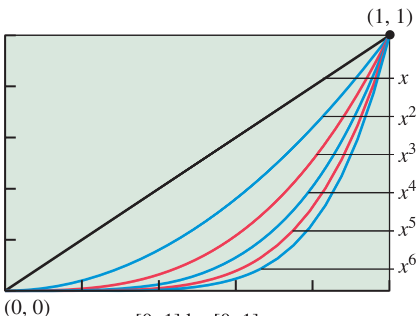
So the zero function and constant functions are monomial functions, but the more typical monomial function is a power function with a positive integer power, which is the degree of the monomial. For example, the basic functions x, x^2, and x^3 are typical monomial functions. It is important to understand the graphs of monomial functions because every polynomial function is either a monomial function or a sum of monomial functions.

In Exploration 1, we take a close look at six basic monomial functions. They have the form x^n for $n = 1, 2, \ldots, 6$. We group them by even and odd powers.

EXPLORATION 1 Comparing Graphs of Monomial Functions

Graph the triplets of functions in the stated windows and explain how the graphs are alike and how they are different. Consider the relevant aspects of analysis from Example 2. Which ordered pairs do all three graphs have in common?

1. $f(x) = x$, $g(x) = x^3$, and $h(x) = x^5$ in the window $[-2.35, 2.35]$ by $[-1.5, 1.5]$, then $[-5, 5]$ by $[-15, 15]$, and finally $[-20, 20]$ by $[-200, 200]$.
2. $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$ in the window $[-1.5, 1.5]$ by $[-0.5, 1.5]$, then $[-5, 5]$ by $[-5, 25]$, and finally $[-15, 15]$ by $[-50, 400]$.



[0, 1] by [0, 1]

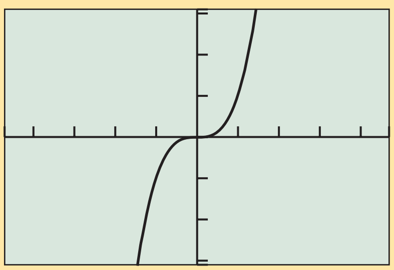
FIGURE 2.11 The graphs of $f(x) = x^n$, $0 \le x \le 1$, for $n = 1, 2, \ldots, 6$.

From Exploration 1 we see that

$f(x) = x^n$ is an even function if n is even and an odd function if n is odd.

Because of this symmetry, it is enough to know the first quadrant behavior of $f(x) = x^n$. Figure 2.11 shows the graphs of $f(x) = x^n$ for $n = 1, 2, \ldots, 6$ in the first quadrant near the origin.

The following conclusions about the basic function $f(x) = x^3$ can be drawn from your investigations in Exploration 1.



[–4.7, 4.7] by [–3.1, 3.1]

FIGURE 2.12 The graph of $f(x) = x^3$.

BASIC FUNCTION The Cubing Function

$f(x) = x^3$
Domain: All reals
Range: All reals
Continuous
Increasing for all x
Symmetric with respect to the origin (an odd function)
Not bounded above or below
No local extrema
No horizontal asymptotes
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} x^3 = -\infty$ and $\lim_{x \to \infty} x^3 = \infty$

EXAMPLE 3 Graphing Monomial Functions

Describe how to obtain the graph of the given function from the graph of $g(x) = x^n$ with the same power n. Sketch the graph by hand and support your answer with a grapher.

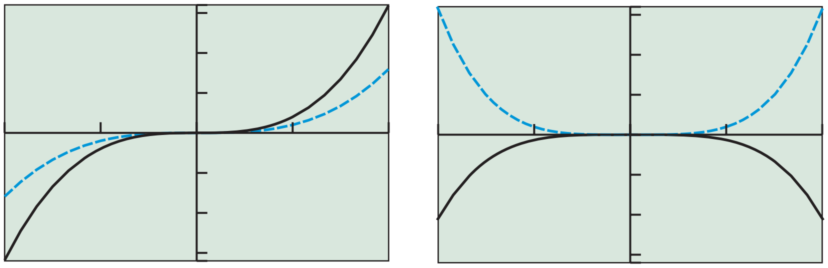
(a) $f(x) = 2x^3$ **(b)** $f(x) = -\dfrac{2}{3}x^4$

SOLUTION

(a) We obtain the graph of $f(x) = 2x^3$ by vertically stretching the graph of $g(x) = x^3$ by a factor of 2. Both are odd functions (Figure 2.13a).

(b) We obtain the graph of $f(x) = -(2/3)x^4$ by vertically shrinking the graph of $g(x) = x^4$ by a factor of 2/3 and then reflecting it across the x-axis. Both are even functions (Figure 2.13b). *Now try Exercise 31.*

We ask you to explore the graphical behavior of power functions of the form x^{-n} and $x^{1/n}$, where n is a positive integer, in Exercise 65.



[–2, 2] by [–16, 16]
(a)

[–2, 2] by [–16, 16]
(b)

FIGURE 2.13 The graphs of (a) $f(x) = 2x^3$ with basic monomial $g(x) = x^3$, and (b) $f(x) = -(2/3)x^4$ with basic monomial $g(x) = x^4$. (Example 3)

Graphs of Power Functions

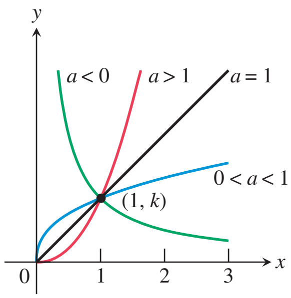
The graphs in Figure 2.14 represent the four shapes that are possible for general power functions of the form $f(x) = kx^a$ for $x \geq 0$. In every case, the graph of f contains $(1, k)$. Those with positive powers also pass through $(0, 0)$. Those with negative exponents are asymptotic to both axes.

When $k > 0$, the graph lies in Quadrant I, but when $k < 0$, the graph is in Quadrant IV.

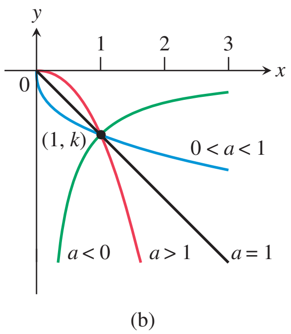
In general, for any power function $f(x) = k \cdot x^a$, one of three following things happens when $x < 0$.

- f is undefined for $x < 0$, as is the case for $f(x) = x^{1/2}$ and $f(x) = x^{\pi}$.
- f is an even function, so f is symmetric about the y-axis, as is the case for $f(x) = x^{-2}$ and $f(x) = x^{2/3}$.
- f is an odd function, so f is symmetric about the origin, as is the case for $f(x) = x^{-1}$ and $f(x) = x^{7/3}$.

Predicting the general shape of the graph of a power function is a two-step process as illustrated in Example 4.



(a)



(b)

FIGURE 2.14 The graphs of $f(x) = k \cdot x^a$ for $x \geq 0$. (a) $k > 0$, (b) $k < 0$.

EXAMPLE 4 Graphing Power Functions $f(x) = k \cdot x^a$

State the values of the constants k and a. Describe the portion of the curve that lies in Quadrant I or IV. Determine whether f is even, odd, or undefined for $x < 0$. Describe the rest of the curve if any. Graph the function to see whether it matches the description.

(a) $f(x) = 2x^{-3}$ **(b)** $f(x) = -0.4x^{1.5}$ **(c)** $f(x) = -x^{0.4}$

SOLUTION

(a) Because $k = 2$ is positive and the power $a = -3$ is negative, the graph passes through $(1, 2)$ and is asymptotic to both axes. The graph is decreasing in the first quadrant. The function f is odd because

$$f(-x) = 2(-x)^{-3} = \frac{2}{(-x)^3} = -\frac{2}{x^3} = -2x^{-3} = -f(x).$$

So its graph is symmetric about the origin. The graph in Figure 2.15a supports all aspects of the description.

(b) Because $k = -0.4$ is negative and the power $a = 1.5 > 1$, the graph contains $(0, 0)$ and passes through $(1, -0.4)$. In the fourth quadrant, it is decreasing. The function f is undefined for $x < 0$ because

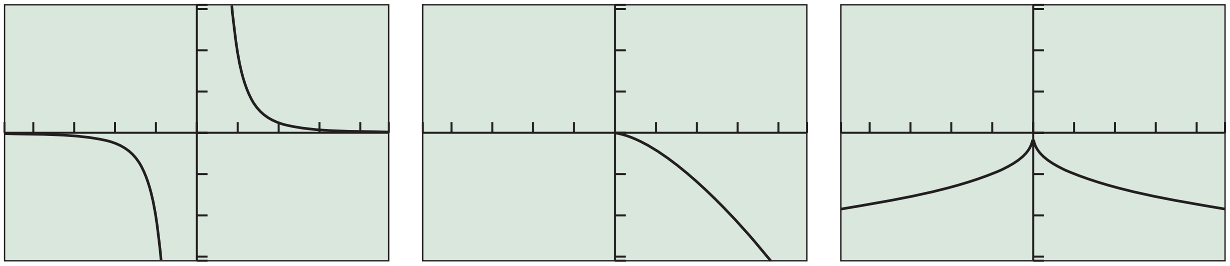
$$f(x) = -0.4x^{1.5} = -\frac{2}{5}x^{3/2} = -\frac{2}{5}(\sqrt{x})^3,$$

and the square root function is undefined for $x < 0$. So the graph of f has no points in Quadrants II or III. The graph in Figure 2.15b matches the description.

(c) Because $k = -1$ is negative and $0 < a < 1$, the graph contains (0, 0) and passes through $(1, -1)$. In the fourth quadrant, it is decreasing. The function f is even because

$$\begin{aligned} f(-x) &= -(-x)^{0.4} = -(-x)^{2/5} = -(\sqrt[5]{-x})^2 = -(-\sqrt[5]{x})^2 \\ &= -(\sqrt[5]{x})^2 = -x^{0.4} = f(x). \end{aligned}$$

So the graph of f is symmetric about the y-axis. The graph in Figure 2.15c fits the description. ***Now try Exercise 43.***



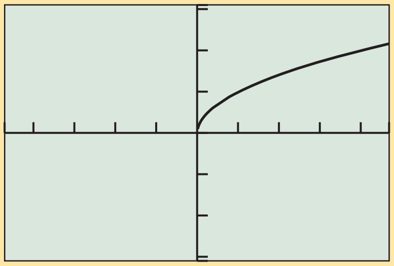
[–4.7, 4.7] by [–3.1, 3.1]
(a)

[–4.7, 4.7] by [–3.1, 3.1]
(b)

[–4.7, 4.7] by [–3.1, 3.1]
(c)

FIGURE 2.15 The graphs of (a) $f(x) = 2x^{-3}$, (b) $f(x) = -0.4x^{1.5}$, and (c) $f(x) = -x^{0.4}$. (Example 4)

The following information about the basic function $f(x) = \sqrt{x}$ follows from the investigation in Exercise 65.



[–4.7, 4.7] by [–3.1, 3.1]

FIGURE 2.16 The graph of $f(x) = \sqrt{x}$.

BASIC FUNCTION The Square Root Function

$f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$
Continuous on $[0, \infty)$
Increasing on $[0, \infty)$
No symmetry
Bounded below but not above
Local minimum at $x = 0$
No horizontal asymptotes
No vertical asymptotes
End behavior: $\lim_{x \to \infty} \sqrt{x} = \infty$

Modeling with Power Functions

Noted astronomer Johannes Kepler (1571–1630) developed three laws of planetary motion that are used to this day. Kepler's Third Law states that the square of the period of orbit T (the time required for one full revolution around the Sun) for each planet is proportional to the cube of its average distance a from the Sun. Table 2.10 gives the relevant data for the six planets that were known in Kepler's time. The distances are given in millions of kilometers, or gigameters (Gm).

Table 2.10 Average Distances and Orbital Periods for the Six Innermost Planets

Planet	Average Distance from Sun (Gm)	Period of Orbit (days)
Mercury	57.9	88
Venus	108.2	225
Earth	149.6	365.2
Mars	227.9	687
Jupiter	778.3	4332
Saturn	1427	10,760

Source: Shupe, Dorr, Payne, Hunsiker, et al., National Geographic Atlas of the World (rev. 6th ed.). Washington, DC: National Geographic Society, 1992, plate 116.

A Bit of History

Example 5 shows the predictive power of a well-founded model. Exercise 67 asks you to find Kepler's elegant form of the equation, $T^2 = a^3$, which he reported in *The Harmony of the World* in 1619.

EXAMPLE 5 Modeling Planetary Data with a Power Function

Use the data in Table 2.10 to obtain a power function model for orbital period as a function of average distance from the Sun. Then use the model to predict the orbital period for Neptune, which is 4497 Gm from the Sun on average.

SOLUTION

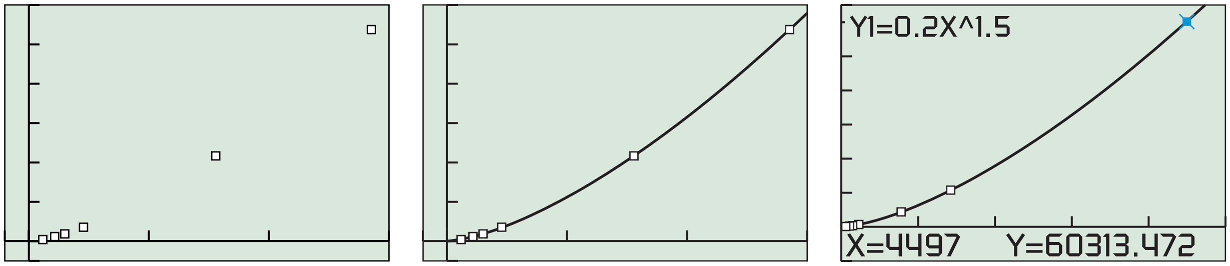
Model

First we make a scatter plot of the data, as shown in Figure 2.17a on page 180. Using power regression, we find the model for the orbital period to be about

$$T(a) \approx 0.20a^{1.5} = 0.20a^{3/2} = 0.20\sqrt{a^3}.$$

Figure 2.17b shows the scatter plot for Table 2.10 together with a graph of the power regression model just found. You can see that the curve fits the data quite well. The coefficient of determination is $r^2 \approx 0.999999912$, indicating an amazingly close fit and supporting the visual evidence.

(continued)



[–100, 1500] by [–1000, 12000]
(a)

[–100, 1500] by [–1000, 12000]
(b)

[0, 5000] by [–10000, 65000]
(c)

FIGURE 2.17 Scatter plot and graphs for Example 5.

Solve Numerically

To predict the orbit period for Neptune we substitute its average distance from the Sun in the power regression model:

$$T(4497) \approx 0.2(4497)^{1.5} \approx 60{,}313$$

Interpret

It takes Neptune about 60,313 days to orbit the Sun, or about 165 years, which is the value given in the *National Geographic Atlas of the World*.

Figure 2.17c reports this result and gives some indication of the relative distances involved. Neptune is much farther from the Sun than the six innermost planets and especially the four closest to the Sun—Mercury, Venus, Earth, and Mars.

Now try Exercise 55.

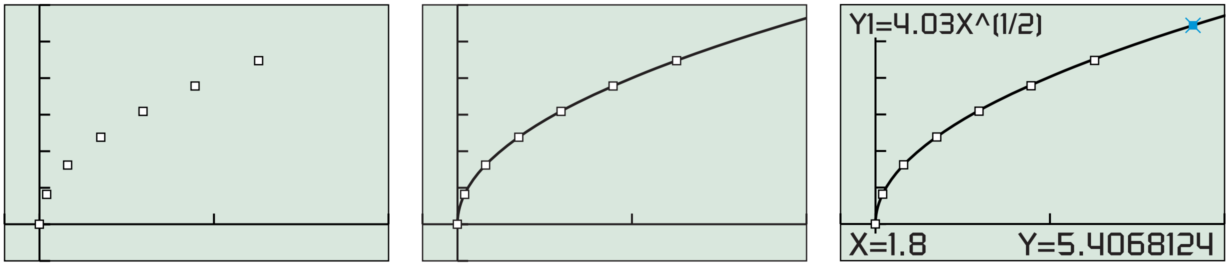
In Example 6, we return to free-fall motion, with a new twist. The data in the table come from the same CBR™ experiment referenced in Example 8 of Section 2.1. This time we are looking at the downward distance (in meters) the ball has traveled since reaching its peak height and its downward speed (in meters per second). It can be shown (see Exercise 68) that free-fall speed is proportional to a power of the distance traveled.

EXAMPLE 6 Modeling Free-Fall Speed Versus Distance

Use the data in Table 2.11 to obtain a power function model for speed p versus distance traveled d. Then use the model to predict the speed of the ball at impact given that impact occurs when $d \approx 1.80$ m.

Table 2.11 Rubber Ball Data from CBR™ Experiment

Distance (m)	Speed (m/s)
0.00000	0.00000
0.04298	0.82372
0.16119	1.71163
0.35148	2.45860
0.59394	3.05209
0.89187	3.74200
1.25557	4.49558



[–0.2, 2] by [–1, 6]
(a)

[–0.2, 2] by [–1, 6]
(b)

[–0.2, 2] by [–1, 6]
(c)

FIGURE 2.18 Scatter plot and graphs for Example 6.

SOLUTION

Model

Figure 2.18a is a scatter plot of the data. Using power regression, we find the model for speed p versus distance d to be about

$$p(d) \approx 4.03d^{0.5} = 4.03d^{1/2} = 4.03\sqrt{d}.$$

(See margin notes.) Figure 2.18b shows the scatter plot for Table 2.11 together with a graph of the power regression equation just found. You can see that the curve fits the data nicely. The coefficient of determination is $r^2 \approx 0.99770$, indicating a close fit and supporting the visual evidence.

Solve Numerically

To predict the speed at impact, we substitute $d \approx 1.80$ into the obtained power regression model:

$$p(1.80) \approx 5.4$$

See Figure 2.18c.

Interpret

The speed at impact is about 5.4 m/sec. This is slightly less than the value obtained in Example 8 of Section 2.1, using a different modeling process for the same experiment. *Now try Exercise 57.*

Why *p*?

We use p for speed to distinguish it from velocity v. Recall that speed is the *absolute value* of velocity.

A Word of Warning

The regression routine traditionally used to compute power function models involves taking logarithms of the data, and therefore, all of the data must be strictly positive numbers. So we must leave out $(0, 0)$ to compute the power regression equation.

QUICK REVIEW 2.2 *(For help, go to Section A.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–6, write the following expressions using only positive integer powers.

1. $x^{2/3}$ **2.** $p^{5/2}$

3. d^{-2} **4.** x^{-7}

5. $q^{-4/5}$ **6.** $m^{-1.5}$

In Exercises 7–10, write the following expressions in the form $k \cdot x^a$ using a single rational number for the power a.

7. $\sqrt{9x^3}$ **8.** $\sqrt[3]{8x^5}$

9. $\sqrt[3]{\dfrac{5}{x^4}}$ **10.** $\dfrac{4x}{\sqrt{32x^3}}$

SECTION 2.2 EXERCISES

In Exercises 1–10, determine whether the function is a power function, given that c, g, k, and π represent constants. For those that are power functions, state the power and constant of variation.

1. $f(x) = -\frac{1}{2}x^5$

2. $f(x) = 9x^{5/3}$

3. $f(x) = 3 \cdot 2^x$

4. $f(x) = 13$

5. $E(m) = mc^2$

6. $KE(v) = \frac{1}{2}kv^5$

7. $d = \frac{1}{2}gt^2$

8. $V = \frac{4}{3}\pi r^3$

9. $I = \frac{k}{d^2}$

10. $F(a) = m \cdot a$

In Exercises 11–16, determine whether the function is a monomial function, given that l and π represent constants. For those that are monomial functions state the degree and leading coefficient. For those that are not, explain why not.

11. $f(x) = -4$

12. $f(x) = 3x^{-5}$

13. $y = -6x^7$

14. $y = -2 \cdot 5^x$

15. $S = 4\pi r^2$

16. $A = lw$

In Exercises 17–22, write the statement as a power function equation. Use k for the constant of variation if one is not given.

17. The area A of an equilateral triangle varies directly as the square of the length s of its sides.

18. The volume V of a circular cylinder with fixed height is proportional to the square of its radius r.

19. The current I in an electrical circuit is inversely proportional to the resistance R, with constant of variation V.

20. Charles's Law states the volume V of an enclosed ideal gas at a constant pressure varies directly as the absolute temperature T.

21. The energy E produced in a nuclear reaction is proportional to the mass m, with the constant of variation being c^2, the square of the speed of light.

22. The speed p of a free-falling object that has been dropped from rest varies as the square root of the distance traveled d, with a constant of variation $k = \sqrt{2g}$.

In Exercises 23–26, write a sentence that expresses the relationship in the formula, using the language of variation or proportion.

23. $w = mg$, where w and m are the weight and mass of an object and g is the constant acceleration due to gravity.

24. $C = \pi D$, where C and D are the circumference and diameter of a circle and π is the usual mathematical constant.

25. $n = c/v$, where n is the refractive index of a medium, v is the velocity of light in the medium, and c is the constant velocity of light in free space.

26. $d = p^2/(2g)$, where d is the distance traveled by a free-falling object dropped from rest, p is the speed of the object, and g is the constant acceleration due to gravity.

In Exercises 27–30, state the power and constant of variation for the function, graph it, and analyze it in the manner of Example 2 of this section.

27. $f(x) = 2x^4$

28. $f(x) = -3x^3$

29. $f(x) = \frac{1}{2}\sqrt[4]{x}$

30. $f(x) = -2x^{-3}$

In Exercises 31–36, describe how to obtain the graph of the given monomial function from the graph of $g(x) = x^n$ with the same power n. State whether f is even or odd. Sketch the graph by hand and support your answer with a grapher.

31. $f(x) = \frac{2}{3}x^4$

32. $f(x) = 5x^3$

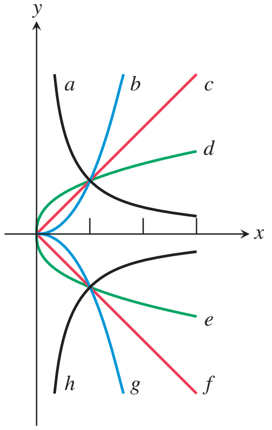
33. $f(x) = -1.5x^5$

34. $f(x) = -2x^6$

35. $f(x) = \frac{1}{4}x^8$

36. $f(x) = \frac{1}{8}x^7$

In Exercises 37–42, match the equation to one of the curves labeled in the figure.



37. $f(x) = -\frac{2}{3}x^4$

38. $f(x) = \frac{1}{2}x^{-5}$

39. $f(x) = 2x^{1/4}$

40. $f(x) = -x^{5/3}$

41. $f(x) = -2x^{-2}$

42. $f(x) = 1.7x^{2/3}$

In Exercises 43–48, state the values of the constants k and a for the function $f(x) = k \cdot x^a$. Before using a grapher, describe the portion of the curve that lies in Quadrant I or IV. Determine whether f is even, odd, or undefined for $x < 0$. Describe the rest of the curve if any. Graph the function to see whether it matches the description.

43. $f(x) = 3x^{1/4}$

44. $f(x) = -4x^{2/3}$

45. $f(x) = -2x^{4/3}$

46. $f(x) = \frac{2}{5}x^{5/2}$

47. $f(x) = \frac{1}{2}x^{-3}$

48. $f(x) = -x^{-4}$

In Exercises 49 and 50, data are given for y as a power function of x. Write an equation for the power function, and state its power and constant of variation.

49.

x	2	4	6	8	10
y	2	0.5	0.222...	0.125	0.08

50.

x	1	4	9	16	25
y	-2	-4	-6	-8	-10

51. Boyle's Law The volume of an enclosed gas (at a constant temperature) varies inversely as the pressure. If the pressure of a 3.46-L sample of neon gas at a temperature of 302 K is 0.926 atm, what would the volume be at a pressure of 1.452 atm if the temperature does not change?

52. Charles's Law The volume of an enclosed gas (at a constant pressure) varies directly as the absolute temperature. If the pressure of a 3.46-L sample of neon gas at a temperature of 302 K is 0.926 atm, what would the volume be at a temperature of 338 K if the pressure does not change?

53. Diamond Refraction Diamonds have the extremely high refraction index of $n = 2.42$ on average over the range of visible light. Use the formula from Exercise 25 and the fact that $c \approx 3.00 \times 10^8$ m/sec to determine the speed of light through a diamond.

54. Windmill Power The power P (in watts) produced by a windmill is proportional to the cube of the wind speed v (in mph). If a wind of 10 mph generates 15 watts of power, how much power is generated by winds of 20, 40, and 80 mph? Make a table and explain the pattern.

55. Keeping Warm For mammals and other warm-blooded animals to stay warm requires quite a bit of energy. Temperature loss is related to surface area, which is related to body weight, and temperature gain is related to circulation, which is related to pulse rate. In the final analysis, scientists have concluded that the pulse rate r of mammals is a power function of their body weight w.

(a) Draw a scatter plot of the data in Table 2.12.

(b) Find the power regression model.

(c) Superimpose the regression curve on the scatter plot.

(d) Use the regression model to predict the pulse rate for a 450-kg horse. Is the result close to the 38 beats/min reported by A. J. Clark in 1927?

Table 2.12 Weight and Pulse Rate of Selected Mammals

Mammal	Body Weight (kg)	Pulse Rate (beats/min)
Rat	0.2	420
Guinea pig	0.3	300
Rabbit	2	205
Small dog	5	120
Large dog	30	85
Sheep	50	70
Human	70	72

Source: A. J. Clark, Comparative Physiology of the Heart. New York: Macmillan, 1927.

56. Even and Odd Functions If n is an integer, $n \geq 1$, prove that $f(x) = x^n$ is an odd function if n is odd and is an even function if n is even.

57. Light Intensity Velma and Reggie gathered the data in Table 2.13 using a 100-watt light bulb and a Calculator-Based Laboratory™ (CBL™) with a light-intensity probe.

(a) Draw a scatter plot of the data in Table 2.13

(b) Find the power regression model. Is the power close to the theoretical value of $a = -2$?

(c) Superimpose the regression curve on the scatter plot.

(d) Use the regression model to predict the light intensity at distances of 1.7 m and 3.4 m.

Table 2.13 Light-Intensity Data for a 100-W Light Bulb

Distance (m)	Intensity (W/m^2)
1.0	7.95
1.5	3.53
2.0	2.01
2.5	1.27
3.0	0.90

Standardized Test Questions

58. True or False The function $f(x) = x^{-2/3}$ is even. Justify your answer.

59. True or False The graph $f(x) = x^{1/3}$ is symmetric about the y-axis. Justify your answer.

In Exercises 60–63, solve the problem without using a calculator.

60. Multiple Choice Let $f(x) = 2x^{-1/2}$. What is the value of $f(4)$?

(A) 1 **(B)** -1 **(C)** $2\sqrt{2}$ **(D)** $\frac{1}{2\sqrt{2}}$ **(E)** 4

61. Multiple Choice Let $f(x) = -3x^{-1/3}$. Which of the following statements is true?

(A) $f(0) = 0$ **(B)** $f(-1) = -3$ **(C)** $f(1) = 1$

(D) $f(3) = 3$ **(E)** $f(0)$ is undefined.

62. Multiple Choice Let $f(x) = x^{2/3}$. Which of the following statements is true?

(A) f is an odd function.

(B) f is an even function.

(C) f is neither an even nor an odd function.

(D) The graph f is symmetric with respect to the x-axis.

(E) The graph f is symmetric with respect to the origin.

63. Multiple Choice Which of the following is the domain of the function $f(x) = x^{3/2}$?

(A) All reals **(B)** $[0,\infty)$ **(C)** $(0,\infty)$

(D) $(-\infty,0)$ **(E)** $(-\infty,0) \cup (0,\infty)$

Explorations

64. Group Activity Rational Powers Working in a group of three students, investigate the behavior of power functions of the form $f(x) = k \cdot x^{m/n}$, where m and n are positive with no factors in common. Have one group member investigate each of the following cases:

- n is even
- n is odd and m is even
- n is odd and m is odd

For each case, decide whether f is even, f is odd, or f is undefined for $x < 0$. Solve graphically and confirm algebraically in a way to convince the rest of your group and your entire class.

65. Comparing the Graphs of Power Functions Graph the functions in the stated windows and explain how the graphs are alike and how they are different. Consider the relevant aspects of analysis from Example 2. Which ordered pairs do all four graphs have in common?

(a) $f(x) = x^{-1}$, $g(x) = x^{-2}$, $h(x) = x^{-3}$, and $k(x) = x^{-4}$ in the windows $[0, 1]$ by $[0, 5]$, $[0, 3]$ by $[0, 3]$, and $[-2, 2]$ by $[-2, 2]$.

(b) $f(x) = x^{1/2}$, $g(x) = x^{1/3}$, $h(x) = x^{1/4}$, and $k(x) = x^{1/5}$ in the windows $[0, 1]$ by $[0, 1]$, $[0, 3]$ by $[0, 2]$, and $[-3, 3]$ by $[-2, 2]$.

Extending the Ideas

66. Writing to Learn Irrational Powers A negative number to an irrational power is undefined. Analyze the graphs of $f(x) = x^{\pi}, x^{1/\pi}, x^{-\pi}, -x^{\pi}, -x^{1/\pi}$, and $-x^{-\pi}$. Prepare a sketch of all six graphs on one set of axes, labeling each of the curves. Write an explanation for why each graph is positioned and shaped as it is.

67. Planetary Motion Revisited Convert the time and distance units in Table 2.10 to the Earth-based units of years and astronomical units using

$$1 \text{ yr} = 365.2 \text{ days and } 1 \text{ AU} = 149.6 \text{ Gm}.$$

Use this "re-expressed" data to obtain a power function model. Show algebraically that this model closely approximates Kepler's equation $T^2 = a^3$.

68. Free Fall Revisited The **speed** p of an object is the absolute value of its velocity v. The distance traveled d by an object dropped from an initial height s_0 with a current height s is given by

$$d = s_0 - s$$

until it hits the ground. Use this information and the free-fall motion formulas from Section 2.1 to prove that

$$d = \frac{1}{2}gt^2,\ p = gt, \text{ and therefore } p = \sqrt{2gd}.$$

Do the results of Example 6 approximate this last formula?

69. Prove that $g(x) = 1/f(x)$ is even if and only if $f(x)$ is even and that $g(x) = 1/f(x)$ is odd if and only if $f(x)$ is odd.

70. Use the results in Exercise 69 to prove that $g(x) = x^{-a}$ is even if and only if $f(x) = x^{a}$ is even and that $g(x) = x^{-a}$ is odd if and only if $f(x) = x^{a}$ is odd.

71. Joint Variation If a variable z varies as the product of the variables x and y, we say z **varies jointly** as x and y, and we write $z = k \cdot x \cdot y$, where k is the constant of variation. Write a sentence that expresses the relationship in each of the following formulas, using the language of joint variation.

(a) $F = m \cdot a$, where F and a are the force and acceleration acting on an object of mass m.

(b) $KE = (1/2)m \cdot v^2$, where KE and v are the kinetic energy and velocity of an object of mass m.

(c) $F = G \cdot m_1 \cdot m_2/r^2$, where F is the force of gravity acting on objects of masses m_1 and m_2 with a distance r between their centers and G is the universal gravitational constant.