CHAPTER 2

Polynomial, Power, and Rational Functions

Humidity and relative humidity are measures used by weather forecasters. Humidity affects our comfort and our health. If humidity is too low, our skin can become dry and cracked, and viruses can live longer. If humidity is too high, it can make warm temperatures feel even warmer, and mold, fungi, and dust mites can live longer. See page 224 to learn how relative humidity is modeled as a rational function.

Chapter 2 Overview

Chapter 1 laid a foundation of the general characteristics of functions, equations, and graphs. In this chapter and the next two, we will explore the theory and applications of specific families of functions. We begin this exploration by studying three interrelated families of functions: polynomial, power, and rational functions. These three families of functions are used in the social, behavioral, and natural sciences.

This chapter includes a thorough study of the theory of polynomial equations. We investigate algebraic methods for finding both real- and complex-number solutions of such equations and relate these methods to the graphical behavior of polynomial and rational functions. The chapter closes by extending these methods to inequalities in one variable.

2.1 Linear and Quadratic Functions and Modeling

What you'll learn about

- Polynomial Functions
- Linear Functions and Their Graphs
- Average Rate of Change
- Linear Correlation and Modeling
- Quadratic Functions and Their Graphs
- Applications of Quadratic Functions

... and why

Many business and economic problems are modeled by linear functions. Quadratic and higher-degree polynomial functions are used in science and manufacturing applications.

Polynomial Functions

Polynomial functions are among the most familiar of all functions.

DEFINITION Polynomial Function

Let n be a nonnegative integer and let $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is a **polynomial function of degree n**. The **leading coefficient** is a_n.

The zero function $f(x) = 0$ is a polynomial function. It has no degree and no leading coefficient.

Polynomial functions are defined and continuous on all real numbers. It is important to recognize whether a function is a polynomial function.

EXAMPLE 1 Identifying Polynomial Functions

Which of the following are polynomial functions? For those that are polynomial functions, state the degree and leading coefficient. For those that are not, explain why not.

(a) $f(x) = 4x^3 - 5x - \dfrac{1}{2}$

(b) $g(x) = 6x^{-4} + 7$

(c) $h(x) = \sqrt{9x^4 + 16x^2}$

(d) $k(x) = 15x - 2x^4$

SOLUTION

(a) f is a polynomial function of degree 3 with leading coefficient 4.

(b) g is not a polynomial function because of the exponent -4.

(c) h is not a polynomial function because it cannot be simplified into polynomial form. Notice that $\sqrt{9x^4 + 16x^2} \neq 3x^2 + 4x$.

(d) k is a polynomial function of degree 4 with leading coefficient -2.

Now try Exercise 1.

The zero function and all constant functions are polynomial functions. Some other familiar functions are also polynomial functions, as shown below.

Polynomial Functions of No and Low Degree

Name	Form	Degree
Zero function	$f(x) = 0$	Undefined
Constant function	$f(x) = a \ (a \neq 0)$	0
Linear function	$f(x) = ax + b \ (a \neq 0)$	1
Quadratic function	$f(x) = ax^2 + bx + c \ (a \neq 0)$	2

We study polynomial functions of degree 3 and higher in Section 2.3. For the remainder of this section, we turn our attention to the nature and uses of linear and quadratic polynomial functions.

Linear Functions and Their Graphs

Linear equations and graphs of lines were reviewed in Sections P.3 and P.4, and some of the examples in Chapter 1 involved linear functions. We now take a closer look at the properties of linear functions.

A **linear function** is a polynomial function of degree 1 and so has the form

$$f(x) = ax + b, \text{ where } a \text{ and } b \text{ are constants and } a \neq 0.$$

If we use m for the leading coefficient instead of a and let $y = f(x)$, then this equation becomes the familiar slope-intercept form of a line:

$$y = mx + b$$

Vertical lines are not graphs of functions because they fail the vertical line test, and horizontal lines are graphs of constant functions. A line in the Cartesian plane is the graph of a linear function if and only if it is a **slant line**, that is, neither horizontal nor vertical. We can apply the formulas and methods of Section P.4 to problems involving linear functions.

EXAMPLE 2 Finding an Equation of a Linear Function

Write an equation for the linear function f such that $f(-1) = 2$ and $f(3) = -2$.

SOLUTION

Solve Algebraically

We seek a line through the points $(-1, 2)$ and $(3, -2)$. The slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 2}{3 - (-1)} = \frac{-4}{4} = -1.$$

Using this slope and the coordinates of $(-1, 2)$ with the point-slope formula, we have

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= -1(x - (-1)) \\ y - 2 &= -x - 1 \\ y &= -x + 1 \end{aligned}$$

Converting to function notation gives us the desired form:

$$f(x) = -x + 1$$

(continued)

Surprising Fact

Not all lines in the Cartesian plane are graphs of linear functions.

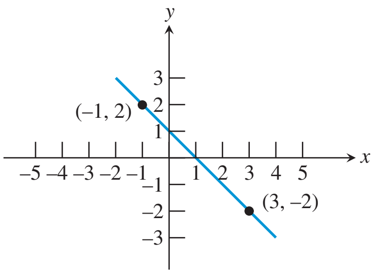


FIGURE 2.1 The graph of $y = -x + 1$ passes through $(-1, 2)$ and $(3, -2)$. (Example 2)

Support Graphically

We can graph $y = -x + 1$ and see that it includes the points $(-1, 2)$ and $(3, -2)$ (Figure 2.1).

Confirm Numerically

Using $f(x) = -x + 1$ we prove that $f(-1) = 2$ and $f(3) = -2$:

$$f(-1) = -(-1) + 1 = 1 + 1 = 2, \text{ and } f(3) = -(3) + 1 = -3 + 1 = -2.$$

Now try Exercise 7.

Average Rate of Change

Another property that characterizes a linear function is its *rate of change*. The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$, $a \neq b$, is

$$\frac{f(b) - f(a)}{b - a}.$$

You are asked to prove the following theorem in Exercise 85.

THEOREM Constant Rate of Change

A function defined on all real numbers is a linear function if and only if it has a constant nonzero average rate of change between any two points on its graph.

Because the average rate of change of a linear function is constant, it is called simply the **rate of change** of the linear function. The slope m in the formula $f(x) = mx + b$ is the rate of change of the linear function. In Exploration 1, we revisit Example 7 of Section P.4 in light of the rate of change concept.

EXPLORATION 1 Modeling Depreciation with a Linear Function

Camelot Apartments bought a \$50,000 building and for tax purposes are depreciating it \$2000 per year over a 25-yr period using straight-line depreciation.

1. What is the rate of change of the value of the building?
2. Write an equation for the value $v(t)$ of the building as a linear function of the time t since the building was placed in service.
3. Evaluate $v(0)$ and $v(16)$.
4. Solve $v(t) = 39{,}000$.

Rate and Ratio

All rates are ratios, whether expressed as miles per hour, dollars per year, or even rise over run.

The rate of change of a linear function is the signed ratio of the corresponding line's rise over run. That is, for a linear function $f(x) = mx + b$,

$$\text{rate of change} = \text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

This formula allows us to interpret the slope, or rate of change, of a linear function numerically. For instance, in Exploration 1 the value of the apartment building fell from \$50,000 to \$0 over a 25-yr period. In Table 2.1 we compute $\Delta y/\Delta x$ for the apartment building's value (in dollars) as a function of time (in years). Because the average rate of change $\Delta y/\Delta x$ is the nonzero constant -2000, the building's value is a linear function of time decreasing at a rate of \$2000/yr.

Table 2.1 Rate of Change of the Value of the Apartment Building in Exploration 1: $y = -2000x + 50{,}000$

x (time)	y (value)	Δx	Δy	$\Delta y/\Delta x$
0	50,000			
		1	−2000	−2000
1	48,000			
		1	−2000	−2000
2	46,000			
		1	−2000	−2000
3	44,000			
		1	−2000	−2000
4	42,000			

In Exploration 1, as in other applications of linear functions, the constant term represents the value of the function for an input of 0. In general, for any function f, $f(0)$ is the **initial value of f**. So for a linear function $f(x) = mx + b$, the constant term b is the initial value of the function. For any polynomial function $f(x) = a_n x^n + \cdots + a_1 x + a_0$, the **constant term** $f(0) = a_0$ is the function's initial value. Finally, the initial value of any function—polynomial or otherwise—is the y-intercept of its graph.

We now summarize what we have learned about linear functions.

Characterizing the Nature of a Linear Function

Point of View	**Characterization**
Verbal	polynomial of degree 1
Algebraic	$f(x) = mx + b \quad (m \neq 0)$
Graphical	slant line with slope m and y-intercept b
Analytical	function with constant nonzero rate of change m: f is increasing if $m > 0$, decreasing if $m < 0$; initial value of the function $= f(0) = b$

Linear Correlation and Modeling

In Section 1.7 we approached modeling from several points of view. Along the way we learned how to use a grapher to create a scatter plot, compute a regression line for a data set, and overlay a regression line on a scatter plot. We touched on the notion of correlation coefficient. We now go deeper into these modeling and regression concepts.

Figure 2.2 on page 162 shows five types of scatter plots. When the points of a scatter plot are clustered along a line, we say there is a **linear correlation** between the quantities represented by the data. When an oval is drawn around the points in the scatter plot, generally speaking, the narrower the oval, the stronger the linear correlation.

When the oval tilts like a line with positive slope (as in Figure 2.2a and b), the data have a **positive linear correlation**. On the other hand, when it tilts like a line with negative slope (as in Figure 2.2d and e), the data have a **negative linear correlation**. Some scatter plots exhibit little or no linear correlation (as in Figure 2.2c), or have nonlinear patterns.

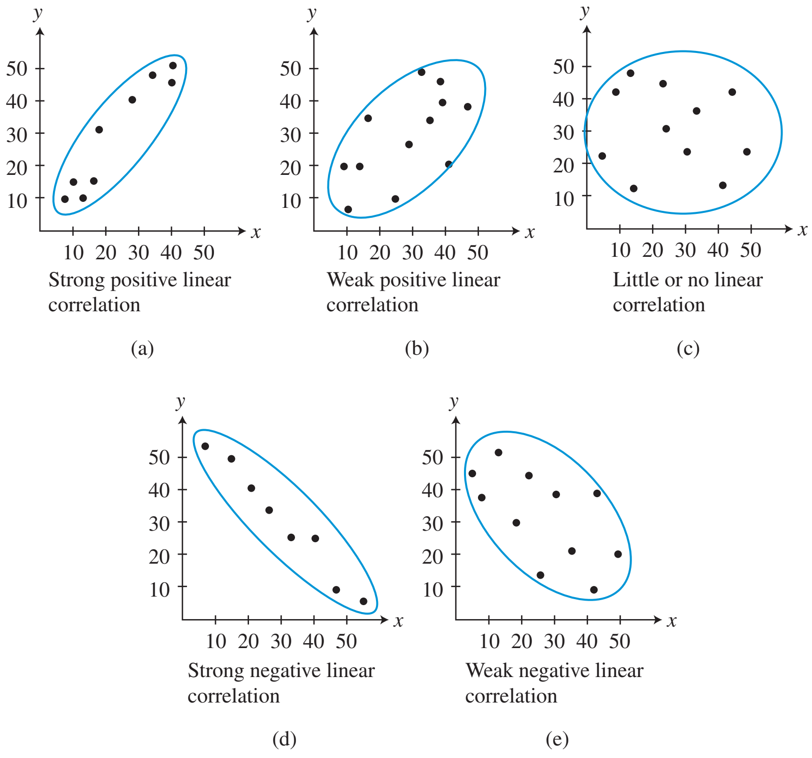


FIGURE 2.2 Five scatter plots and the types of linear correlation they suggest.

Correlation vs. Causation

Correlation does not imply causation. Two variables can be strongly correlated, but that does not necessarily mean that one *causes* the other.

A number that measures the strength and direction of the linear correlation of a data set is the **(linear) correlation coefficient, *r***.

Properties of the Correlation Coefficient, *r*

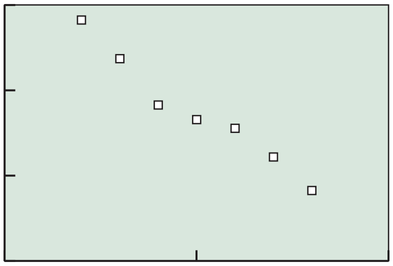
1. $-1 \le r \le 1$.
2. When $r > 0$, there is a positive linear correlation.
3. When $r < 0$, there is a negative linear correlation.
4. When $|r| \approx 1$, there is a strong linear correlation.
5. When $r \approx 0$, there is weak or no linear correlation.

Correlation informs the modeling process by giving us a measure of goodness of fit. Good modeling practice, however, demands that we have a theoretical reason for selecting a model. In business, for example, fixed cost is modeled by a constant function. (Otherwise, the cost would not be fixed.)

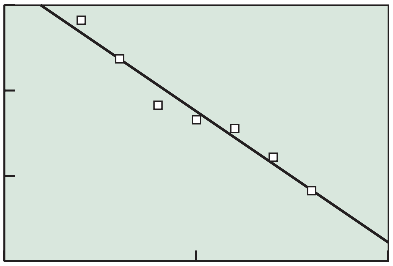
In economics, a linear model is often used for the demand for a product as a function of its price. For instance, suppose Twin Pixie, a large supermarket chain, conducts a market analysis on its store brand of doughnut-shaped oat breakfast cereal. The chain sets various prices for its 15-oz box at its different stores over a period of time. Then, using these data, the Twin Pixie researchers predict the weekly sales at the entire chain of stores for each price and obtain the data shown in Table 2.2.

Table 2.2 Weekly Sales Data Based on Marketing Research

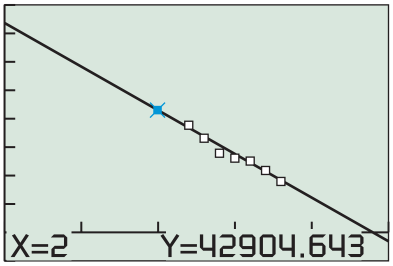
Price per Box	Boxes Sold
\$2.40	38,320
\$2.60	33,710
\$2.80	28,280
\$3.00	26,550
\$3.20	25,530
\$3.40	22,170
\$3.60	18,260



[2, 4] by [10000, 40000]
(a)



[2, 4] by [10000, 40000]
(b)



[0, 5] by [–10000, 80000]
(c)

FIGURE 2.3 Scatter plot and regression line graphs for Example 3.

EXAMPLE 3 Modeling and Predicting Demand

Use the data in Table 2.2 to write a linear model for demand (in boxes sold per week) as a function of the price per box (in dollars). Describe the strength and direction of the linear correlation. Then use the model to predict weekly cereal sales if the price is dropped to \$2.00 or raised to \$4.00 per box.

SOLUTION

Model

We enter the data and obtain the scatter plot shown in Figure 2.3a. It appears that the data have a strong negative correlation.

We then find the linear regression model to be approximately

$$y = -15{,}358.93x + 73{,}622.50,$$

where x is the price per box of cereal and y the number of boxes sold.

Figure 2.3b shows the scatter plot for Table 2.2 together with a graph of the regression line. You can see that the line fits the data fairly well. The correlation coefficient of $r \approx -0.98$ supports this visual evidence.

Solve Graphically

Our goal is to predict the weekly sales for prices of \$2.00 and \$4.00 per box. Using the value feature of the grapher, as shown in Figure 2.3c, we see that y is about 42,900 when x is 2. In a similar manner we could find that $y \approx 12{,}190$ when x is 4.

Interpret

If Twin Pixie drops the price for its store brand of doughnut-shaped oat breakfast cereal to \$2.00 per box, demand will rise to about 42,900 boxes per week. On the other hand, if they raise the price to \$4.00 per box, demand will drop to around 12,190 boxes per week. *Now try Exercise 49.*

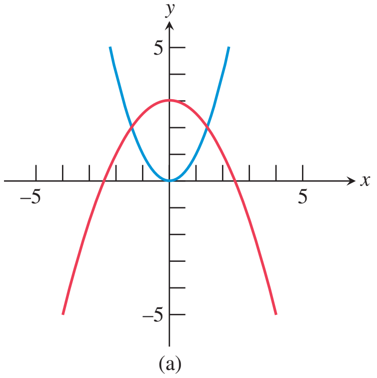
We summarize for future reference the analysis used in Example 3.

Regression Analysis

1. Enter and plot the data (scatter plot).
2. Find the regression model that fits the problem situation.
3. Superimpose the graph of the regression model on the scatter plot, and observe the fit.
4. Use the regression model to make the predictions called for in the problem.

Quadratic Functions and Their Graphs

A **quadratic function** is a polynomial function of degree 2. Recall from Section 1.3 that the graph of the squaring function $f(x) = x^2$ is a parabola. We will see that the graph of every quadratic function is an upward- or downward-opening parabola. This is because the graph of any quadratic function can be obtained from the graph of the squaring function $f(x) = x^2$ by a sequence of translations, reflections, stretches, and shrinks.



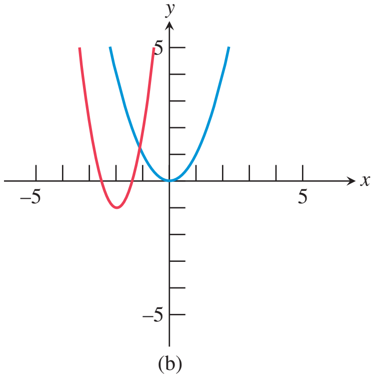


FIGURE 2.4 The graph of $f(x) = x^2$ (blue) shown with (a) $g(x) = -(1/2)x^2 + 3$ and (b) $h(x) = 3(x + 2)^2 - 1$. (Example 4)

EXAMPLE 4 Transforming the Squaring Function

Describe how to transform the graph of $f(x) = x^2$ into the graph of the given function. Sketch its graph by hand.

(a) $g(x) = -(1/2)x^2 + 3$ **(b)** $h(x) = 3(x + 2)^2 - 1$

SOLUTION

(a) The graph of $g(x) = -(1/2)x^2 + 3$ is obtained by vertically shrinking the graph of $f(x) = x^2$ by a factor of 1/2, reflecting the resulting graph across the x-axis, and translating the reflected graph up 3 units (Figure 2.4a).

(b) The graph of $h(x) = 3(x + 2)^2 - 1$ is obtained by vertically stretching the graph of $f(x) = x^2$ by a factor of 3 and translating the resulting graph left 2 units and down 1 unit (Figure 2.4b). ***Now try Exercise 19.***

The graph of $f(x) = ax^2$, $a > 0$, is an upward-opening parabola. When $a < 0$, its graph is a downward-opening parabola. Regardless of the sign of a, the y-axis is the line of symmetry for the graph of $f(x) = ax^2$. The line of symmetry for a parabola is its **axis of symmetry**, or **axis** for short. The point on the parabola that intersects its axis is the **vertex** of the parabola. Because the graph of a quadratic function is always an upward- or downward-opening parabola, its vertex is always the lowest or highest point of the parabola. The vertex of $f(x) = ax^2$ is always the origin, as seen in Figure 2.5.

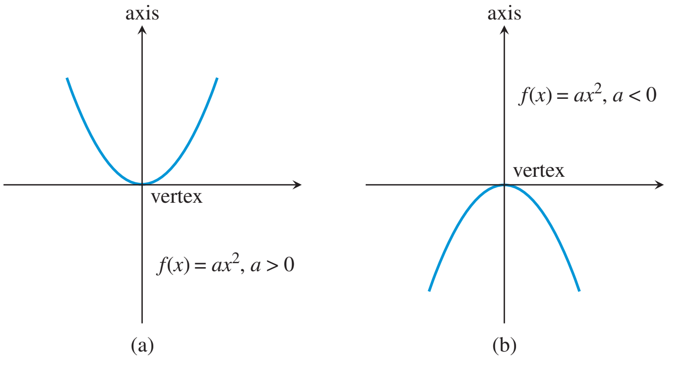


FIGURE 2.5 The graph $f(x) = ax^2$ for (a) $a > 0$ and (b) $a < 0$.

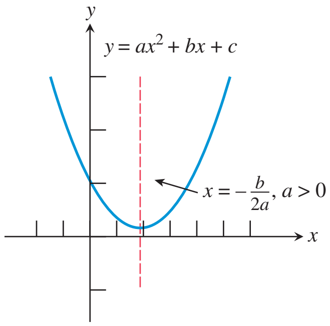
Expanding $f(x) = a(x - h)^2 + k$ and comparing the resulting coefficients with the **standard quadratic form** $ax^2 + bx + c$, where the powers of x are arranged in descending order, we can obtain formulas for h and k.

$$
\begin{aligned}
f(x) &= a(x - h)^2 + k \\
&= a(x^2 - 2hx + h^2) + k && \text{Expand } (x - h)^3. \\
&= ax^2 + (-2ah)x + (ah^2 + k) && \text{Distributive property} \\
&= ax^2 + bx + c && \text{Let } b = -2ah \text{ and } c = ah^2 + k.
\end{aligned}
$$

Because $b = -2ah$ and $c = ah^2 + k$ in the last line above, $h = -b/(2a)$ and $k = c - ah^2$. Using these formulas, any quadratic function $f(x) = ax^2 + bx + c$ can be rewritten in the form

$$f(x) = a(x - h)^2 + k.$$

This *vertex form* for a quadratic function makes it easy to identify the vertex and axis of the graph of the function, and to sketch the graph.



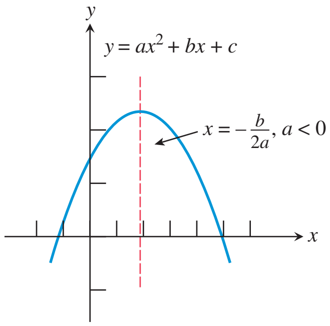


FIGURE 2.6 The vertex is at $x = -b/(2a)$, which therefore also describes the axis of symmetry. (a) When $a > 0$, the parabola opens upward. (b) When $a < 0$, the parabola opens downward.

Vertex Form of a Quadratic Function

Any quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, can be written in the **vertex form**

$$f(x) = a(x - h)^2 + k.$$

The graph of f is a parabola with vertex (h, k) and axis $x = h$, where $h = -b/(2a)$ and $k = c - ah^2$. If $a > 0$, the parabola opens upward, and if $a < 0$, it opens downward (Figure 2.6).

The formula $h = -b/(2a)$ is useful for locating the vertex and axis of the parabola associated with a quadratic function. To help you remember it, notice that $-b/(2a)$ is part of the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(Cover the radical term.) You need not remember $k = c - ah^2$ because you can use $k = f(h)$ instead, as illustrated in Example 5.

EXAMPLE 5 Finding the Vertex and Axis of a Quadratic Function

Use the vertex form of a quadratic function to find the vertex and axis of the graph of $f(x) = 6x - 3x^2 - 5$. Rewrite the equation in vertex form.

SOLUTION

Solve Algebraically

The standard polynomial form of f is

$$f(x) = -3x^2 + 6x - 5.$$

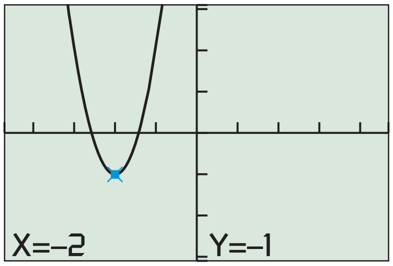
So $a = -3$, $b = 6$, and $c = -5$, and the coordinates of the vertex are

$$h = -\frac{b}{2a} = -\frac{6}{2(-3)} = 1 \text{ and}$$
$$k = f(h) = f(1) = -3(1)^2 + 6(1) - 5 = -2.$$

The equation of the axis is $x = 1$, the vertex is $(1, -2)$, and the vertex form of f is

$$f(x) = -3(x - 1)^2 + (-2).$$

Now try Exercise 27.



[–4.7, 4.7] by [–3.1, 3.1]

FIGURE 2.7 The graphs of $f(x) = 3x^2 + 12x + 11$ and $y = 3(x + 2)^2 - 1$ appear to be identical. The vertex $(-2, -1)$ is highlighted. (Example 6)

EXAMPLE 6 Using Algebra to Describe the Graph of a Quadratic Function

Use completing the square to describe the graph of $f(x) = 3x^2 + 12x + 11$. Support your answer graphically.

SOLUTION

Solve Algebraically

$$\begin{aligned} f(x) &= 3x^2 + 12x + 11 \\ &= 3(x^2 + 4x) + 11 && \text{Factor 3 from the } x\text{-terms.} \\ &= 3(x^2 + 4x + (\ ) - (\ )) + 11 && \text{Prepare to complete the square.} \\ &= 3(x^2 + 4x + (2^2) - (2^2)) + 11 && \text{Complete the square.} \\ &= 3(x^2 + 4x + 4) - 3(4) + 11 && \text{Distribute the 3.} \\ &= 3(x + 2)^2 - 1 \end{aligned}$$

The graph of f is an upward-opening parabola with vertex $(-2, -1)$, axis of symmetry $x = -2$. (The x-intercepts are $x = -2 \pm \sqrt{3}/3$, or about -2.577 and -1.423.)

Support Graphically

The graph in Figure 2.7 supports these results. *Now try Exercise 33.*

We now summarize what we know about quadratic functions.

Characterizing the Nature of a Quadratic Function

Point of View	Characterization
Verbal	polynomial of degree 2
Algebraic	$f(x) = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$ $(a \neq 0)$
Graphical	parabola with vertex (h, k) and axis $x = h$; opens upward if $a > 0$, opens downward if $a < 0$; initial value $= y$-intercept $= f(0) = c$; $x\text{-intercepts} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Applications of Quadratic Functions

In economics, when demand is linear, revenue is quadratic. Example 7 illustrates this by extending the Twin Pixie model of Example 3.

EXAMPLE 7 Predicting Maximum Revenue

Use the model $y = -15{,}358.93x + 73{,}622.50$ from Example 3 to develop a model for the weekly revenue generated by doughnut-shaped oat breakfast cereal sales. Determine the maximum revenue and how to achieve it.

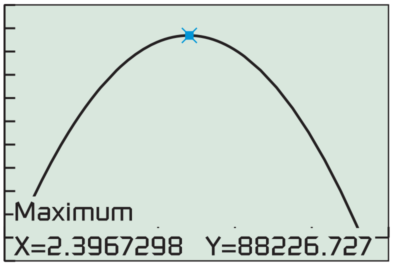
SOLUTION

Model

Revenue can be found by multiplying the price per box, x, by the number of boxes sold, y. So the revenue is given by

$$R(x) = x \cdot y = -15{,}358.93x^2 + 73{,}622.50x,$$

a quadratic model.



[0, 5] by [–10000, 100000]

FIGURE 2.8 The revenue model for Example 7.

Solve Graphically

In Figure 2.8, we find a maximum of about 88,227 occurs when x is about 2.40.

Interpret

To maximize revenue, Twin Pixie should set the price for its store brand of doughnut-shaped oat breakfast cereal at \$2.40 per box. Based on the model, this will yield a weekly revenue of about \$88,227. ***Now try Exercise 55.***

Recall that the average rate of change of a linear function is constant. In Exercise 78 you will see that the average rate of change of a quadratic function is not constant.

In calculus you will study not only average rate of change but also *instantaneous rate of change*. Such instantaneous rates include velocity and acceleration, which we now begin to investigate.

Since the time of Galileo Galilei (1564–1642) and Isaac Newton (1642–1727), the vertical motion of a body in free fall has been well understood. The *vertical velocity* and *vertical position* (*height*) of a free-falling body (as functions of time) are classical applications of linear and quadratic functions.

Vertical Free-Fall Motion

The **height** s and **vertical velocity** v of an object in free fall are given by

$$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0 \quad \text{and} \quad v(t) = -gt + v_0,$$

where t is time (in seconds), $g \approx 32$ ft/sec^2 ≈ 9.8 m/sec^2 is the **acceleration due to gravity**, v_0 is the *initial vertical velocity* of the object, and s_0 is its *initial height*.

These formulas disregard air resistance, and the two values given for g are valid at sea level. We apply these formulas in Example 8, and will use them from time to time throughout the rest of the book.

The data in Table 2.3 were collected in Boone, North Carolina (about 1 km above sea level), using a Calculator-Based Ranger™ (CBR™) and a 15-cm rubber air-filled ball. The CBR™ was placed on the floor face up. The ball was thrown upward above the CBR™, and it landed directly on the face of the device.

Table 2.3 Rubber Ball Data from CBR™

Time (sec)	Height (m)
0.0000	1.03754
0.1080	1.40205
0.2150	1.63806
0.3225	1.77412
0.4300	1.80392
0.5375	1.71522
0.6450	1.50942
0.7525	1.21410
0.8600	0.83173

EXAMPLE 8 Modeling Vertical Free-Fall Motion

Use the data in Table 2.3 to write models for the height and vertical velocity of the rubber ball. Then use these models to predict the maximum height of the ball and its vertical velocity when it hits the face of the CBR™.

SOLUTION

Model

First we make a scatter plot of the data, as shown in Figure 2.9a. Using quadratic regression, we find the model for the height of the ball to be about

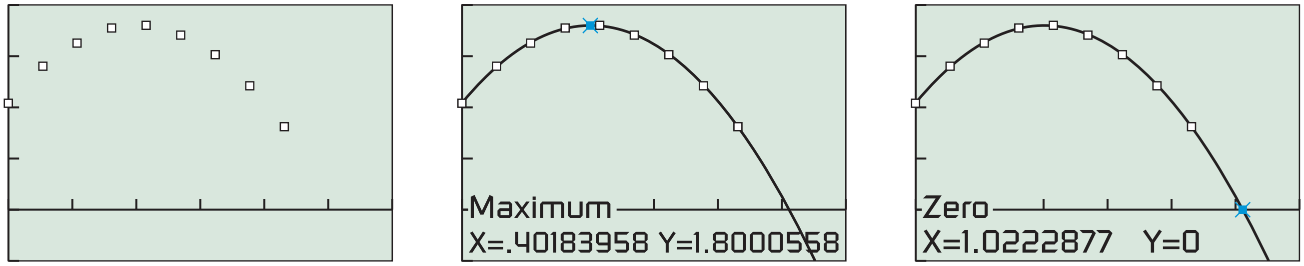
$$s(t) = -4.676t^2 + 3.758t + 1.045,$$

with $R^2 \approx 0.999$, indicating an excellent fit.

Our free-fall theory says the leading coefficient of -4.676 is $-g/2$, giving us a value for $g \approx 9.352$ m/sec^2, which is a bit less than the theoretical value of 9.8 m/sec^2. We also obtain $v_0 \approx 3.758$ m/sec. So the model for vertical velocity becomes

$$v(t) = -gt + v_0 \approx -9.352t + 3.758.$$

(continued)



[0, 1.2] by [–0.5, 2.0]
(a)

[0, 1.2] by [–0.5, 2.0]
(b)

[0, 1.2] by [–0.5, 2.0]
(c)

FIGURE 2.9 Scatter plot and graph of height versus time for Example 8.

Solve Graphically and Numerically

The maximum height is the maximum value of $s(t)$, which occurs at the vertex of its graph. We can see from Figure 2.9b that the vertex has coordinates of about $(0.402, 1.800)$.

In Figure 2.9c, to determine when the ball hits the face of the CBR™, we calculate the positive-valued zero of the height function, which is $t \approx 1.022$. We turn to our linear model to compute the vertical velocity at impact:

$$v(1.022) = -9.352(1.022) + 3.758 \approx -5.80 \text{ m/sec}$$

Interpret

The maximum height the ball achieved was about 1.80 m above the face of the CBR™. The ball's downward rate is about 5.80 m/sec when it hits the CBR™.

The curve in Figure 2.9b appears to fit the data extremely well, and $R^2 \approx 0.999$. You may have noticed, however, that Table 2.3 contains the ordered pair $(0.4300, 1.80392)$ and that $1.80392 > 1.800$, which is the maximum shown in Figure 2.9b. So, even though our model is theoretically based and an excellent fit to the data, it is not a perfect model. Despite its imperfections, the model provides accurate and reliable predictions about the CBR™ experiment. ***Now try Exercise 63.***

Reminder

Recall from Section 1.7 that R^2 is the coefficient of determination, which measures goodness of fit.

QUICK REVIEW 2.1 *(For help, go to Sections A.2. and P.4)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–2, write an equation in slope-intercept form for a line with the given slope m and y-intercept b.

1. $m = 8,\quad b = 3.6$ **2.** $m = -1.8,\quad b = -2$

In Exercises 3–4, write an equation for the line containing the given points. Graph the line and points.

3. $(-2, 4)$ and $(3, 1)$ **4.** $(1, 5)$ and $(-2, -3)$

In Exercises 5–8, expand the expression.

5. $(x + 3)^2$ **6.** $(x - 4)^2$

7. $3(x - 6)^2$ **8.** $-3(x + 7)^2$

In Exercises 9–10, factor the trinomial.

9. $2x^2 - 4x + 2$ **10.** $3x^2 + 12x + 12$

SECTION 2.1 EXERCISES

In Exercises 1–6, determine which are polynomial functions. For those that are, state the degree and leading coefficient. For those that are not, explain why not.

1. $f(x) = 3x^{-5} + 17$ **2.** $f(x) = -9 + 2x$

3. $f(x) = 2x^5 - \frac{1}{2}x + 9$ **4.** $f(x) = 13$

5. $h(x) = \sqrt[3]{27x^3 + 8x^6}$ **6.** $k(x) = 4x - 5x^2$

In Exercises 7–12, write an equation for the linear function f satisfying the given conditions. Graph $y = f(x)$.

7. $f(-5) = -1$ and $f(2) = 4$

8. $f(-3) = 5$ and $f(6) = -2$

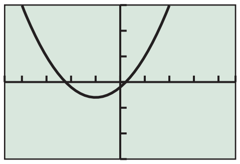
9. $f(-4) = 6$ and $f(-1) = 2$

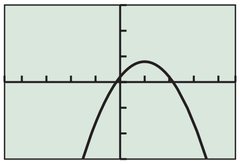
10. $f(1) = 2$ and $f(5) = 7$

11. $f(0) = 3$ and $f(3) = 0$

12. $f(-4) = 0$ and $f(0) = 2$

In Exercises 13–18, match a graph to the function. Explain your choice.


(a)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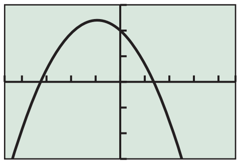
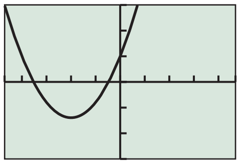

(c)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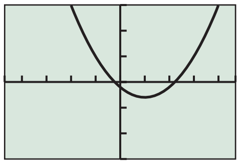
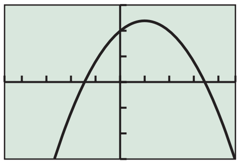

(e)


(f)

13. $f(x) = 2(x + 1)^2 - 3$ **14.** $f(x) = 3(x + 2)^2 - 7$

15. $f(x) = 4 - 3(x - 1)^2$ **16.** $f(x) = 12 - 2(x - 1)^2$

17. $f(x) = 2(x - 1)^2 - 3$ **18.** $f(x) = 12 - 2(x + 1)^2$

In Exercises 19–22, describe how to transform the graph of $f(x) = x^2$ into the graph of the given function. Sketch each graph by hand.

19. $g(x) = (x - 3)^2 - 2$ **20.** $h(x) = \frac{1}{4}x^2 - 1$

21. $g(x) = \frac{1}{2}(x + 2)^2 - 3$ **22.** $h(x) = -3x^2 + 2$

In Exercises 23–26, find the vertex and axis of the graph of the function.

23. $f(x) = 3(x - 1)^2 + 5$ **24.** $g(x) = -3(x + 2)^2 - 1$

25. $f(x) = 5(x - 1)^2 - 7$ **26.** $g(x) = 2(x - \sqrt{3})^2 + 4$

In Exercises 27–32, find the vertex and axis of the graph of the function. Rewrite the equation for the function in vertex form.

27. $f(x) = 3x^2 + 5x - 4$ **28.** $f(x) = -2x^2 + 7x - 3$

29. $f(x) = 8x - x^2 + 3$ **30.** $f(x) = 6 - 2x + 4x^2$

31. $g(x) = 5x^2 + 4 - 6x$ **32.** $h(x) = -2x^2 - 7x - 4$

In Exercises 33–38, use completing the square to describe the graph of each function. Support your answers graphically.

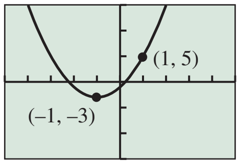
33. $f(x) = x^2 - 4x + 6$ **34.** $g(x) = x^2 - 6x + 12$

35. $f(x) = 10 - 16x - x^2$ **36.** $h(x) = 8 + 2x - x^2$

37. $f(x) = 2x^2 + 6x + 7$ **38.** $g(x) = 5x^2 - 25x + 12$

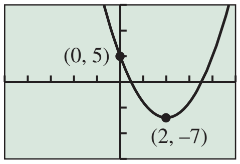
In Exercises 39–42, write an equation for the parabola shown, using the fact that one of the given points is the vertex.

39.



[–5, 5] by [–15, 15]

40.



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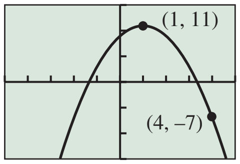

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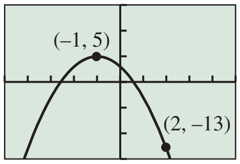


[–5, 5] by [–15, 15]

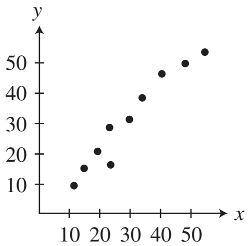
In Exercises 43 and 44, write an equation for the quadratic function whose graph contains the given vertex and point.

43. Vertex $(1, 3)$, point $(0, 5)$

44. Vertex $(-2, -5)$, point $(-4, -27)$

In Exercises 45–48, describe the strength and direction of the linear correlation.

45.



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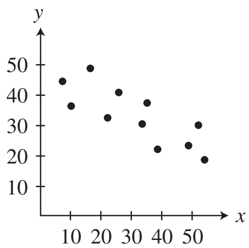
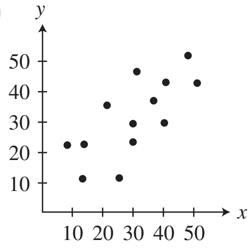
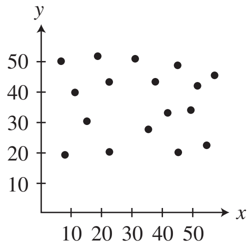


47.



48.



49. Comparing Age and Weight A group of male children were weighed. Their ages and weights are recorded in Table 2.4.

Table 2.4 Children's Age and Weight

Age (months)	Weight (pounds)
18	23
20	25
24	24
26	32
27	33
29	29
34	35
39	39
42	44

(a) Draw a scatter plot of these data.

(b) **Writing to Learn** Describe the strength and direction of the correlation between age and weight.

50. Life Expectancy Table 2.5 shows the average number of additional years a U.S. citizen is expected to live for various ages.

Table 2.5 U.S. Life Expectancy

Age (years)	Remaining Life Expectancy (years)
10	68.5
20	58.8
30	49.3
40	39.9
50	30.9
60	22.5
70	15.1

Source: United States Life Tables, 2004. National Vital Statistics Reports, December, 2007.

(a) Draw a scatter plot of these data.

(b) **Writing to Learn** Describe the strength and direction of the correlation between age and life expectancy.

51. Straight-Line Depreciation Mai Lee bought a computer for her home office and depreciated it over 5 years using the straight-line method. If its initial value was \$2350, what is its value 3 years later?

52. Costly Doll Making Patrick's doll-making business has weekly fixed costs of \$350. If the cost for materials is \$4.70 per doll and his total weekly costs average \$500, about how many dolls does Patrick make each week?

53. Table 2.6 shows the average hourly compensation of U.S. production workers for several years. Let x be the number of years since 1970, so that $x = 5$ stands for 1975, and so forth.

Table 2.6 Production Worker Earnings

Year	Hourly Compensation (dollars)
1975	4.73
1985	8.74
1995	11.65
2005	16.13

Source: U.S. Bureau of Labor Statistics as reported in The World Almanac and Book of Facts 2009.

(a) **Writing to Learn** Find the linear regression model for the data. What does the slope in the regression model represent?

(b) Use the linear regression model to predict the production worker average hourly compensation in the year 2015.

54. Finding Maximum Area Among all the rectangles whose perimeters are 100 ft, find the dimensions of the one with maximum area.

55. Determining Revenue The per unit price p (in dollars) of a popular toy when x units (in thousands) are produced is modeled by the function

$$\text{price} = p = 12 - 0.025x.$$

The revenue (in thousands of dollars) is the product of the price per unit and the number of units (in thousands) produced. That is,

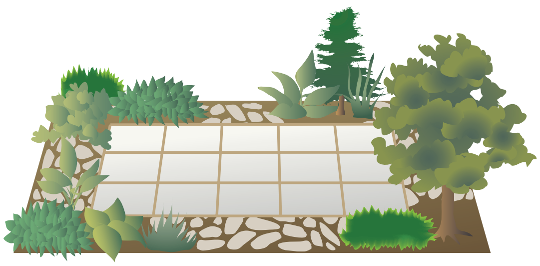
$$\text{revenue} = xp = x(12 - 0.025x).$$

(a) State the dimensions of a viewing window that shows a graph of the revenue model for producing 0 to 100,000 units.

(b) How many units should be produced if the total revenue is to be \$1,000,000?

56. Finding the Dimensions of a Painting A large painting in the style of Rubens is 3 ft longer than it is wide. If the wooden frame is 12 in. wide, the area of the picture and frame is 208 ft^2, find the dimensions of the painting.

57. Using Algebra in Landscape Design Julie Stone designed a rectangular patio that is 25 ft by 40 ft. This patio is surrounded by a terraced strip of uniform width planted with small trees and shrubs. If the area A of this terraced strip is 504 ft^2, find the width x of the strip.



58. Management Planning The Welcome Home apartment rental company has 1600 units available, of which 800 are currently rented at \$300 per month. A market survey indicates that each \$5 decrease in monthly rent will result in 20 new leases.

(a) Determine a function $R(x)$ that models the total rental income realized by Welcome Home, where x is the number of \$5 decreases in monthly rent.

(b) Find a graph of $R(x)$ for rent levels between \$175 and \$300 (that is, $0 \leq x \leq 25$) that clearly shows a maximum for $R(x)$.

(c) What rent will yield Welcome Home the maximum monthly income?

59. Group Activity Beverage Business The Sweet Drip Beverage Co. sells cans of soda pop in machines. It finds that sales average 26,000 cans per month when the cans sell for 50¢ each. For each nickel increase in the price, the sales per month drop by 1000 cans.

(a) Determine a function $R(x)$ that models the total revenue realized by Sweet Drip, where x is the number of \$0.05 increases in the price of a can.

(b) Find a graph of $R(x)$ that clearly shows a maximum for $R(x)$.

(c) How much should Sweet Drip charge per can to realize the maximum revenue? What is the maximum revenue?

60. Group Activity Sales Manager Planning Jack was named District Manager of the Month at the Athens Wire Co. due to his hiring study. It shows that each of the 30 salespersons he supervises average \$50,000 in sales each month, and that for each additional salesperson he would hire, the average sales would decrease \$1000 per month. Jack concluded his study by suggesting a number of salespersons that he should hire to maximize sales. What was that number?

61. Free-Fall Motion As a promotion for the Houston Astros downtown ballpark, a competition is held to see who can throw a baseball the highest from the front row of the upper deck of seats, 83 ft above field level. The winner throws the ball with an initial vertical velocity of 92 ft/sec and it lands on the infield grass.

(a) Find the maximum height of the baseball.

(b) How much time is the ball in the air?

(c) Determine its vertical velocity when it hits the ground.

62. Baseball Throwing Machine The Sandusky Little League uses a baseball throwing machine to help train 10-year-old players to catch high pop-ups. It throws the baseball straight up with an initial velocity of 48 ft/sec from a height of 3.5 ft.

(a) Find an equation that models the height of the ball t seconds after it is thrown.

(b) What is the maximum height the baseball will reach? How many seconds will it take to reach that height?

63. Fireworks Planning At the Bakersville Fourth of July celebration, fireworks are shot by remote control into the air from a pit that is 10 ft below the earth's surface.

(a) Find an equation that models the height of an aerial bomb t seconds after it is shot upward with an initial velocity of 80 ft/sec. Graph the equation.

(b) What is the maximum height above ground level that the aerial bomb will reach? How many seconds will it take to reach that height?

64. Landscape Engineering In her first project after being employed by Land Scapes International, Becky designs a decorative water fountain that will shoot water to a maximum height of 48 ft. What should be the initial velocity of each drop of water to achieve this maximum height? (*Hint*: Use a grapher and a guess-and-check strategy.)

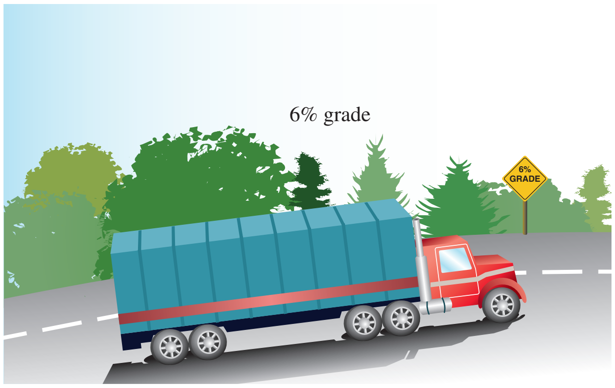
65. Patent Applications Using quadratic regression on the data in Table 2.7, predict the year when the number of patent applications reached 450,000. Let $x = 0$ stand for 1980, $x = 10$ for 1990, and so forth.

Table 2.7 U.S. Patent Applications

Year	Applications (thousands)
1980	113.0
1990	176.7
1995	228.8
1998	261.4
1999	289.5
2000	315.8
2001	346.6
2002	357.5
2003	367.0

Source: U.S. Census Bureau, Statistical Abstract of the United States, 2004–2005 (124th ed., Washington, D.C., 2004).

66. Highway Engineering Interstate 70 west of Denver, Colorado, has a section posted as a 6% grade. This means that for a horizontal change of 100 ft there is a 6-ft vertical change.



(a) Find the slope of this section of the highway.

(b) On a highway with a 6% grade what is the horizontal distance required to climb 250 ft?

(c) A sign along the highway says 6% grade for the next 7 mi. Estimate how many feet of vertical change there are along those 7 mi. (There are 5280 ft in 1 mile.)

67. A group of female children were weighed. Their ages and weights are recorded in Table 2.8.

Table 2.8 Children's Ages and Weights

Age (months)	Weight (pounds)
19	22
21	23
24	25
27	28
29	31
31	28
34	32
38	34
43	39

(a) Draw a scatter plot of the data.

(b) Find the linear regression model.

(c) Interpret the slope of the linear regression equation.

(d) Superimpose the regression line on the scatter plot.

(e) Use the regression model to predict the weight of a 30-month-old girl.

68. Table 2.9 shows the median U.S. income of women (in 2007 dollars) for selected years. Let x be the number of years since 1940.

Table 2.9 Median Income of Women in the United States (in 2007 dollars)

Year	Median Income ($)
1950	7,165
1960	7,726
1970	10,660
1980	11,787
1990	15,486
2000	19,340
2007	20,922

Source: U.S. Census Bureau as reported in The World Almanac and Book of Facts 2009.

(a) Find the linear regression model for the data.

(b) Use it to predict the median U.S. female income in 2015.

In Exercises 69–70, complete the analysis for the given Basic Function.

69. Analyzing a Function

BASIC FUNCTION
The Identity Function

$f(x) = x$
Domain:
Range:
Continuity:
Increasing-decreasing behavior:
Symmetry:
Boundedness:
Local extrema:
Horizontal asymptotes:
Vertical asymptotes:
End behavior:

70. Analyzing a Function

BASIC FUNCTION
The Squaring Function

$f(x) = x^2$
Domain:
Range:
Continuity:
Increasing-decreasing behavior:
Symmetry:
Boundedness:
Local extrema:
Horizontal asymptotes:
Vertical asymptotes:
End behavior:

Standardized Test Questions

71. True or False The initial value of $f(x) = 3x^2 + 2x - 3$ is 0. Justify your answer.

72. True or False The graph of the function $f(x) = x^2 - x + 1$ has no x-intercepts. Justify your answer.

In Exercises 73–76, you may use a graphing calculator to solve the problem.

In Exercises 73 and 74, $f(x) = mx + b$, $f(-2) = 3$, and $f(4) = 1$.

73. Multiple Choice What is the value of m?

(A) 3 **(B)** -3 **(C)** -1 **(D)** 1/3 **(E)** $-1/3$

74. Multiple Choice What is the value of b?

(A) 4 **(B)** 11/3 **(C)** 7/3 **(D)** 1 **(E)** $-1/3$

In Exercises 75 and 76, let $f(x) = 2(x + 3)^2 - 5$.

75. Multiple Choice What is the axis of symmetry of the graph of f?

(A) $x = 3$ **(B)** $x = -3$ **(C)** $y = 5$
(D) $y = -5$ **(E)** $y = 0$

76. Multiple Choice What is the vertex of f?

(A) $(0, 0)$ **(B)** $(3, 5)$ **(C)** $(3, -5)$
(D) $(-3, 5)$ **(E)** $(-3, -5)$

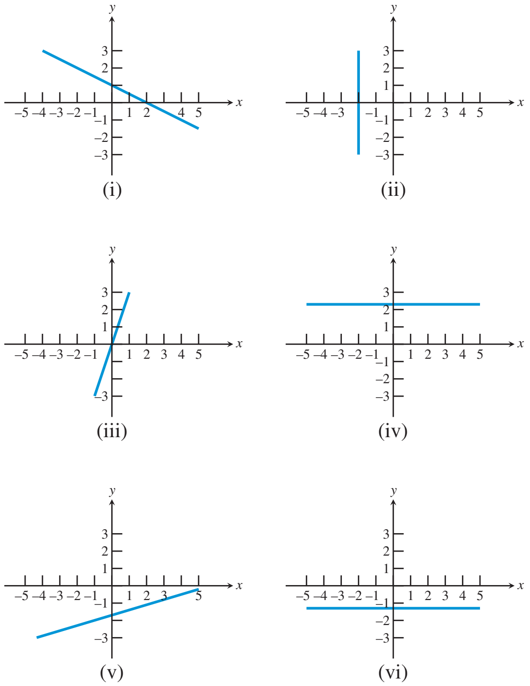
Explorations

77. Writing to Learn Identifying Graphs of Linear Functions

(a) Which of the lines graphed on the next page are graphs of linear functions? Explain.

(b) Which of the lines graphed on the next page are graphs of functions? Explain.

(c) Which of the lines graphed on the next page are not graphs of functions? Explain.



78. Average Rate of Change Let $f(x) = x^2$, $g(x) = 3x + 2$, $h(x) = 7x - 3$, $k(x) = mx + b$, and $l(x) = x^3$.

(a) Compute the average rate of change of f from $x = 1$ to $x = 3$.

(b) Compute the average rate of change of f from $x = 2$ to $x = 5$.

(c) Compute the average rate of change of f from $x = a$ to $x = c$.

(d) Compute the average rate of change of g from $x = 1$ to $x = 3$.

(e) Compute the average rate of change of g from $x = 1$ to $x = 4$.

(f) Compute the average rate of change of g from $x = a$ to $x = c$.

(g) Compute the average rate of change of h from $x = a$ to $x = c$.

(h) Compute the average rate of change of k from $x = a$ to $x = c$.

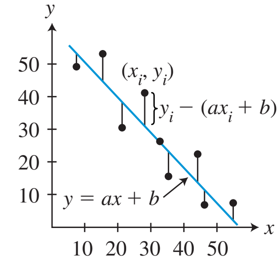
(i) Compute the average rate of change of l from $x = a$ to $x = c$.

Extending the Ideas

79. Minimizing Sums of Squares The linear regression line is often called the **least-square lines** because it minimizes the sum of the squares of the **residuals**, the differences between actual y values and predicted y values:

$$\textit{residual} = y_i - (ax_i + b),$$

where (x_i, y_i) are the given data pairs and $y = ax + b$ is the regression equation, as shown in the figure.



Use these definitions to explain why the regression line obtained from reversing the ordered pairs in Table 2.2 is not the inverse of the function obtained in Example 3.

80. Median-Median Line Read about the median-median line by going to the Internet, your grapher owner's manual, or a library. Then use the following data set to complete this problem.

$$\{(2, 8), (3, 6), (5, 9), (6, 8), (8, 11), (10, 13), (12, 14), (15, 4)\}$$

(a) Draw a scatter plot of the data.

(b) Find the linear regression equation and graph it.

(c) Find the median-median line equation and graph it.

(d) **Writing to Learn** For these data, which of the two lines appears to be the line of better fit? Why?

81. Suppose $b^2 - 4ac > 0$ for the equation $ax^2 + bx + c = 0$.

(a) Prove that the sum of the two solutions of this equation is $-b/a$.

(b) Prove that the product of the two solutions of this equation is c/a.

82. Connecting Algebra and Geometry Prove that the axis of the graph of $f(x) = (x - a)(x - b)$ is $x = (a + b)/2$, where a and b are real numbers.

83. Connecting Algebra and Geometry Identify the vertex of the graph of $f(x) = (x - a)(x - b)$, where a and b are any real numbers.

84. Connecting Algebra and Geometry Prove that if x_1 and x_2 are real numbers and are zeros of the quadratic function $f(x) = ax^2 + bx + c$, then the axis of the graph of f is $x = (x_1 + x_2)/2$.

85. Prove the Constant Rate of Change Theorem, which is stated on page 160.